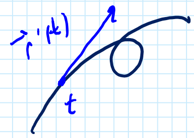
Math 261, Lecture 7, 9/5/18

Today: §13.3, Next: §13.4

Recap: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ a curve

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$$, parametric form

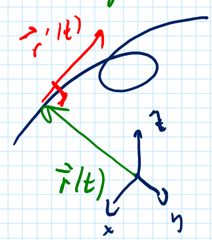


$\frac{d}{dt}\vec{r}(t) = \vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$ tangent vector to curve at time t

$T(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ <u>unit</u> tangent vector at time t

↳ directional component

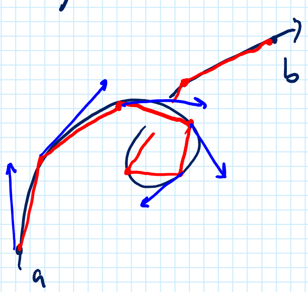
$|\vec{r}(t)| = c$, then $\vec{r}'(t) \perp \vec{r}(t)$ [See Example 4 p. 858]



§13.3 Arc Length and Curvature

$\vec{r}(t)$

$a \leq t \leq b$



|line segment Δt|

$\approx |\vec{r}'(t)| \Delta t$

$y = f(x)$

$\frac{|\Delta y|}{|\Delta x|} \sim |f'(x)|$

Q: How to compute distance traveled (length)?

Let $\Delta t \to 0$

"add lengths of small segments"

$$L = \int_a^b |\vec{r}'(t)|\, dt$$

$\vec{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$s(l) = \int_a^l |r'(u)|\, du \quad \leftarrow \text{fxn of } l$$

FTC $\quad \frac{ds}{dt} = |\vec{r}\,'(t)|$

Ex. Cylindrical Spiral $\quad x = \cos(t)$, $y = \sin(t)$, $z = t$

Compute $s(t)$ $\quad a = 0$

$$\frac{dx}{dt} = -\sin(t), \quad \frac{dy}{dt} = \cos(t), \quad \frac{dz}{dt} = 1$$

$$s(l) = \int_0^l \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2}\, dt$$

$$= \int_0^l \sqrt{\sin(t)^2 + \cos(t)^2 + 1}\, dt$$

$$= \int_0^l \sqrt{2}\, dt = \sqrt{2}\, l \qquad \frac{ds}{dt} = \sqrt{2}$$

This shows this is a constant velocity parametrization

Ex. $\quad x = t^2$, $y = 1$, $z = t^3$ $\quad$ Compute $s(t)$, $a = 0$

$x = \cos(t^3)$, $y = \sin(t^3)$, $z = t^3$ $\quad$ parametrizes same curve but "velocity" is not constant

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 0, \quad \frac{dz}{dt} = 3t^2$$

$$s(l) = \int_0^l \sqrt{(2t)^2 + (0)^2 + (3t^2)^2}\, dt$$

$$= \int_0^l \sqrt{4t^2 + 9t^4}\, dt \qquad \text{Factor out } t^2$$

$$= \int_0^l \sqrt{(4 + 9t^2)t^2}\, dt$$

$$= \int_0^{\ell} \sqrt{(4+9t^2)t^2}\, dt$$

$$= \int_0^{\ell} \left(\sqrt{4+9t^2}\right) t\, dt \qquad u = 4+9t^2$$

$$du = 18t\, dt$$

$$\tfrac{1}{18}\, du = t\, dt$$

$$= \int_4^{4+9\ell^2} \tfrac{1}{18}\sqrt{u}\, du$$

Curvature = "change in direction per unit <u>length</u>"

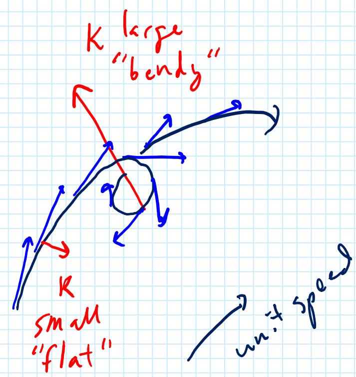
<u>NOT</u> per unit time

* Does not depend on parametrization *

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

↑ baby kappa

Use this if $|\vec{r}'(t)|$ is complicated



Ex. $x = \cos(t)$
$y = \sin(t)$
$z = t$

What is $|\vec{r}'(t)|$?

$|r'(t)| = \sqrt{2}$ by 1^{st} example

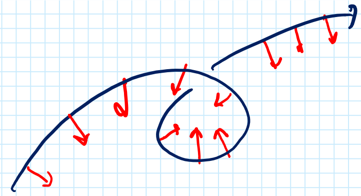
$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -\sin(t), \cos(t), 1\rangle}{\sqrt{2}}$$

$$\vec{T}'(t) = \frac{\langle -\cos(t), -\sin(t), 0\rangle}{\sqrt{2}}$$

$$|\vec{T}'(t)| = \sqrt{\left(-\frac{\cos(t)}{\sqrt{2}}\right)^2 + \left(-\frac{\sin(t)}{\sqrt{2}}\right)^2 + \left(\frac{0}{\sqrt{2}}\right)^2} = \sqrt{\tfrac{1}{2}\cos(t)^2 + \tfrac{1}{2}\sin(t)^2} = \frac{1}{\sqrt{2}}$$

Answer $\kappa = \dfrac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \dfrac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \dfrac{1}{2}$

Normal to the curve $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$



Normal points "into" the curve

Fact : Since $|\vec{T}(t)| = 1$ we have that $\vec{T}(t) \perp \vec{N}(t)$

Ex. $x = t^2$
$y = 1$
$z = t^3$

Find κ and $\vec{N}(t)$

$\vec{r}\,'(t) = \langle 2t, 0, 3t^2 \rangle$ $\quad |\vec{r}\,'(t)| = (4t^2 + 9t^4)^{1/2}$

by second example

$\vec{r}\,''(t) = \langle 2, 0, 6t \rangle$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} i & j & k \\ 2t & 0 & 3t^2 \\ 2 & 0 & 6t \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 3t^2 \\ 0 & 6t \end{vmatrix} i - \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} j + \begin{vmatrix} 2t & 0 \\ 2 & 0 \end{vmatrix} k$$

0 0 0 0

$$= 0i - (12t^2 - 6t^2)j + 0k$$

$$= \langle 0, -6t^2, 0 \rangle$$

So $|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{0^2 + (-6t^2)^2 + 0^2} = \sqrt{36t^4} = 6t^2$

Thus $\kappa = \dfrac{6t^2}{\left(\sqrt{4t^2 + 9t^4}\right)^3} = \dfrac{6t^2}{(4t^2 + 9t^4)^{3/2}}$

Compare even in this simple case to computing

$$\frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}\ !$$

$$= \left\langle \frac{-36t^4}{\left(\sqrt{4t^2+9t^4}\right)^3},\ 0,\ \frac{12t^3}{\left(\sqrt{4t^2+9t^4}\right)^3} \right\rangle$$