Math 261, Lecture 8, 9/7/18

Today: §13.4 beginning to Ex 5. Next: §14.1

Recap: $\vec{r}(t) = f(t)\,i + g(t)\,j + h(t)\,k$

Arc Length $a \le t \le b$

$$L = \int_a^b |r'(t)|\,dt$$

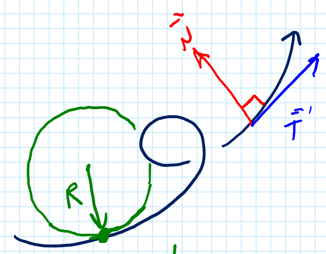
Describe "shape" of curve:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$ unit tangent vector

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$ curvature

curvature is a number

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$$ unit normal vector



$K \sim \frac{1}{R}$

§13.4 Velocity and Acceleration

$$\vec{r}(t) = f(t)\,i + g(t)\,j + h(t)\,k$$

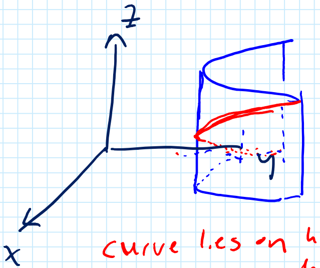
$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t) = \vec{r}\,'(t)$$ ← velocity vector

$$|\vec{v}(t)| = |\vec{r}\,'(t)|$$ ← speed (number)

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$ ← acceleration vector

Ex. $\vec{r}(t) = \langle \sinh(t), \cosh(t), t \rangle$

Find velocity, speed, accel.



curve lies on hyperbolic cylinder

$$\vec{v}(t) = \langle \cosh(t), \sinh(t), 1 \rangle$$

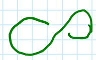
$$|\vec{v}(t)| = \sqrt{\underbrace{\cosh(t)^2 + \sinh(t)^2 + 1^2}_{=}}$$

Use $\cosh(t)^2 - \sinh(t)^2 = 1$

$$= \sqrt{2\cosh(t)^2} \overset{\cosh(t)^2}{=} \sqrt{2}\cosh(t)$$ Note $\cosh(t) \geq 1$ positive

$$\vec{a}(t) = \vec{v}\,'(t) = \langle \sinh(t), \cosh(t), 0 \rangle$$

Ex. $\begin{cases} x = \cos(t) \\ y = \sin(t) \\ z = 3 + \sin(t)^2 \end{cases}$



See lecture 5
curve is intersection of
cylinder $x^2+y^2=1$ with
hyperbolic paraboloid $2x^2-3y^2-z=1$

Answer

$$\vec{v}(t) = -\sin(t)\,i + \cos(t)\,j + \big(\underbrace{2\sin(t)\cos(t)}_{\sin(2t)}\big)\,k$$

$$= -\sin(t)\,i + \cos(t)\,j + \sin(2t)\,k$$

$$|\vec{v}(t)| = \sqrt{(-\sin(t))^2 + \cos(t)^2 + \sin(2t)^2}$$
$$= \sqrt{1+\sin(2t)^2}$$

$$\vec{a}(t) = -\cos(t)\,i - \sin(t)\,j + 2\cos(2t)\,k$$

Ex. $\vec{a}(t) = (12t)\,i - 4\,j$

$\vec{v}(0) = \langle 1, 0, -3 \rangle$

$\vec{r}(0) = \langle 0, -1, 1 \rangle$

Find $\vec{r}(t)$

"Initial Value Problem"

By FTC
$$\vec{v}(t) = \int a(t)\,dt$$
$$= \int [(12t)\,i - 4\,j]\,dt$$
$$= 6t^2\,i - 4t\,j + \vec{C}$$
$$\vec{C} = c_1 i + c_2 j + c_3 k$$

$$i - 3k = \vec{v}(0) = \langle 0,0,0 \rangle + \vec{c} = c_1 i + c_2 j + c_3 k$$

$\vec{C} = \langle 1, 0, -3 \rangle$

$$\boxed{\vec{v}(t) = (6t^2+1)\,i \;-4t\,j \;-3\,k}$$

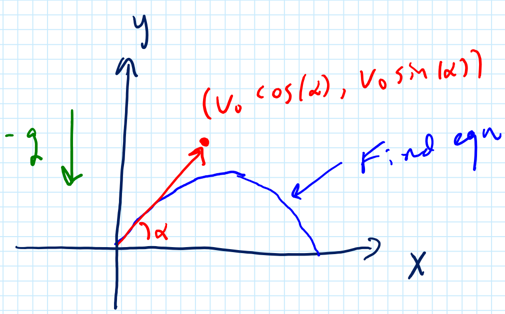
FTC $\quad \vec{r}(t) = \int \vec{v}(t)\,dt = (2t^3+t)\,i - 2t^2\,j - 3t\,k + \vec{C}$

initial conditions

$$\langle 0,-1,1 \rangle = \vec{r}(0) = \langle 0,0,0 \rangle + \vec{C} \rightsquigarrow \vec{C} = \langle 0,-1,1 \rangle$$

$$\boxed{\vec{r}(t) = (2t^3+t)\,i + (-2t^2-1)\,j + (-3t+1)\,k}$$

Ex.



object is thrown
initial speed v_0
at angle α

$$\vec{v}(0) = v_0\cos(\alpha)\,i + v_0\sin(\alpha)\,j$$

$$\vec{a}(t) = -g\,j$$

$$\vec{v}(t) = c_1\,i + (c_2 - gt)\,j$$

$$\vec{v}(0) = v_0\cos(\alpha)\,i + v_0\sin(\alpha)\,j = c_1\,i + c_2\,j$$

$$\vec{v}(t) = v_0\cos(\alpha)\,i + (v_0\sin(\alpha) - gt)\,j$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt$$

$$= v_0\cos(\alpha)t\,i + \left(v_0\sin(\alpha)t - \frac{g}{2}t^2\right)j + \vec{C}$$

initial condition $\vec{r}(0) = 0i + 0j \rightsquigarrow \vec{C} = 0$

$$\vec{r}(t) = v_0\cos(\alpha)t\,i + \left(v_0\sin(\alpha)t - \frac{g}{2}t^2\right)j \quad \underline{\text{or}} \quad \begin{cases} x(t) = v_0\cos(\alpha)t \\ y(t) = v_0\sin(\alpha)t - \frac{g}{2}t^2 \end{cases}$$

What is the time when the object hits the ground

j component $= 0$

$$v_0 \sin(\alpha) t - \frac{g}{2} t^2 = 0$$

$$t\left(v_0 \sin(\alpha) - \frac{g}{2} t \right) = 0$$

$t = 0$ or

$$\boxed{t = \frac{2 v_0 \sin(\alpha)}{g}}$$

Position hits the ground? Find the x component at the time when the object hits the ground

$$x(t) = v_0 \cos(\alpha) t$$

Plug in $t = \dfrac{2 v_0 \sin(\alpha)}{g}$

$$x = v_0 \cos(\alpha) \left[\frac{2 v_0 \sin(\alpha)}{g} \right] = \frac{2 v_0^2 \sin(\alpha) \cos(\alpha)}{g}$$

or $\boxed{x = \dfrac{v_0^2 \sin(2\alpha)}{g}}$

Note this obtains its maximum when $\sin(2\alpha) = 1$ or $2\alpha = \frac{\pi}{2} \rightsquigarrow \alpha = \frac{\pi}{4}$

Thus an object travels farthest when launched at a $45°$ angle!