SECTION: ____________

Midterm 1

MA266, Sinclair, Fall 2015

September 29, 2015

NAME: Solutions

I have read and understood the Student Honor Code, and this exam reflects my unwavering commitment to the principles of academic integrity and honesty expressed therein.

SCORES:

1. ____________ /10

2. ____________ /10

3. ____________ /10

4. ____________ /10

5. ____________ /10

Total: ____________ /50

TO RECEIVE FULL CREDIT, PLEASE SHOW ALL WORK.

Problem 1 (10 points total) *Circle all correct responses.*

(A) Circle all methods which apply for the equation

$$t^3y' + 3t^2y + 2ty^2 = 0.$$

(a) Linear; (b) Separable; (c) Homogenous; (d) Bernoulli; (e) Exact.

(B) Determine the number of stable and unstable equilibria for

$$(y+1)y' = (y-2)(9-y^2).$$

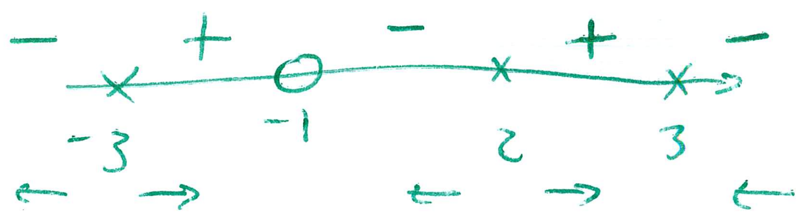
(a) 2 stable, 2 unstable; (b) 2 stable, 1 unstable; (c) 1 stable, 2 unstable;
(d) 3 stable; 1 unstable; (e) 1 stable, 1 unstable.

(C) Use Euler's method with $h = 0.5$ to estimate $y(2)$ for the initial value problem

$$y' = 3 + 2t - y, \; y(1) = -1.$$

(a) 7/2; (b) 4; (c) -7/2; (d) 5; (e) 5/2.

(B) $y' = \dfrac{(y-2)(9-y^2)}{y+1}$ 3 roots $y = 2, 3, -3$
1 asymptote $y = -1$



(c) $y(1) = y_0 = -1$, $t_0 = 1$

$y_1 = -1 + (3 + 2\cdot 1 - (-1)).5 = -1 + 3 = 2$, $t_1 = 3/2$

$y(2) \approx y_2 = 2 + (3 + 2(\tfrac{3}{2}) - 2)\cdot .5 = 2 + 2 = 4$, $t_2 = 2$

Problem 2 (10 points total) *Solve the following. (5 points each.)*

(A) Sketch the region in the ty-plane of all points (t_0, y_0) *where the initial value problem*

$$ty' = t(4y - t^2)^{1/2} + 1, \;\; y(t_0) = y_0$$

has a unique solution.

(B) Determine the maximal interval where a solution to the initial value problem

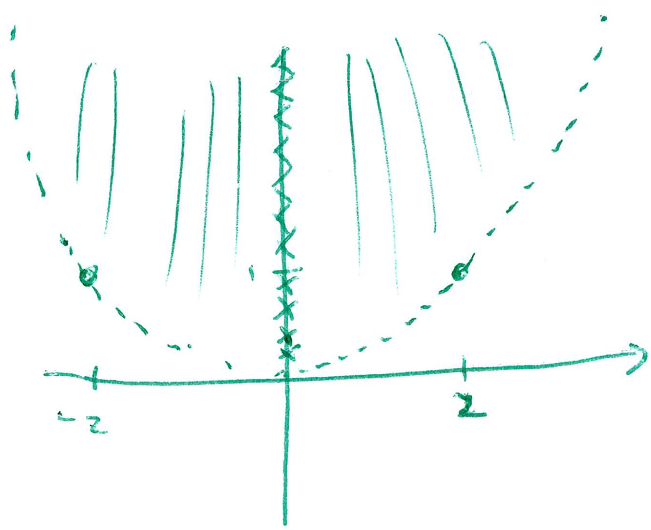
$$(4 - e^t)y' + \frac{2t-5}{t-8}y = 3\sqrt{t-1}, \;\; y(6) = \pi$$

can be shown to exist.

(A) $y' = \overbrace{(4y - t^2)^{1/2} + \frac{1}{t}}^{\text{"}f(x,y)\text{"}} \rightsquigarrow \begin{array}{l} 4y - t^2 \geq 0 \\ t \neq 0 \end{array}$

$\frac{\partial}{\partial y} f(x,y) = \frac{2}{(4y-t^2)^{1/2}} \rightsquigarrow 4y - t^2 > 0$

domains of both $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ need to be considered by Thm 2.4.2.



(B) Thm 2.4.1

$y' + \frac{(2t-5)}{(t-8)(4-e^t)} y = \frac{3\sqrt{t-1}}{4-e^t}$

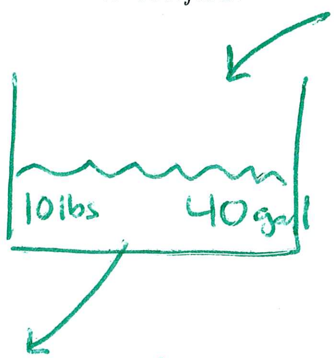
$t \neq 8, \; t \geq 1, \; e^t \neq 4 \rightsquigarrow t \neq \ln(4)$

$1 < \ln(4) < \underset{t_0}{\overset{\shortparallel}{6}} < 8$

so $(\ln 4, 8)$

Problem 3 (10 points total.) *A tank with a capacity of 100 gal originlly contains 40 gal of water with 10 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 2 gal/min, and the mixture is allowed to flow out of the tank at a rate of 1 gal/min. Find the amount and concentration of salt in the tank at any time prior to when the solution begins to overflow.*

2 gal/min in
1 lb/gal mix.



1 gal/min out

$V = V(t) =$ liquid in tank at time $t = 40+t$ gal.

$Q = Q(t)$ quantity of salt (lbs.)

$$\frac{dQ}{dt} = \text{rate salt in} - \text{rate salt out}$$

$$= 2\,\frac{\text{gal}}{\text{min}} \times 1\,\frac{\text{lb}}{\text{gal}} - 1\,\frac{\text{gal}}{\text{min}}\,\underbrace{\frac{Q}{40+t}}_{\substack{\text{concentration}\\ \text{of salt}\\ \text{time } t}}$$

$$\frac{dQ}{dt} + \frac{1}{40+t}Q = 2$$

1st order linear

$$\mu(t) = \exp\left(\int \frac{1}{40+t}\,dt\right)$$
$$= \exp(\ln(40+t))$$
$$= 40+t$$

$$\frac{d}{dt}\Big[(40+t)Q\Big] = 80+2t$$

$$Q(t) = \frac{80t+t^2+C}{40+t}$$

$Q(0) = 10 \rightsquigarrow C = 400$

$$Q(t) = \frac{80t+t^2+400}{40+t}$$

$$C(t) = \frac{80t+t^2+400}{(40+t)^2}$$

Concentration.

Problem 4 (10 points total) *Solve the following differential equations. Solutions may be written in implicit form. (5 points each.)*

(A)

$$\frac{dy}{dx} = \frac{3-\cos(x)}{2\sin(y)-3}$$

(B)

$$t\frac{dy}{dt} + ty = 1 - 2y.$$

(A) separable $\int 2\sin(y) - 3 \frac{dy}{dx}\,dx = \int 3 - \cos(x)\,dx$

$\rightsquigarrow -2\cos(y) - 3y = 3x - \sin(x) + C$

or $-2\cos(y) - 3y - 3x + \sin(x) = C$

(B) linear (collect terms) $ty' + (t+2)y = 1$

standard form $y' + \left(1 + \frac{2}{t}\right)y = \frac{1}{t}$

$\mu(t) = \exp\left(\int \left(1 + \frac{2}{t}\right)dt\right) = \exp(t + 2\ln(t))$

$= \exp(t + \ln(t^2)) = t^2 e^t$

$\frac{d}{dt}[t^2 e^t y] = te^t$ so,

$t^2 e^t y = \int te^t\,dt = te^t - e^t + C$

$u = t \quad v = e^t$

$u' = 1 \quad v' = e^t$

Problem 5 (10 points total) *Use the best method the solve the following initial value problems. Solutions may be written in implicit form. (5 points each.)*

(A)

$$t^2y' + ty - 2y^2 = 0, \; y(1) = 2.$$

(B)

$$y' = \frac{3x^2 - 4y^2}{8xy + 8y^2}, \; y(0) = 1.$$

(A) Bernoulli: $y' + \frac{1}{t}y = \frac{2}{t^2}y^2$

$\frac{1}{y^2}y' + \frac{1}{t}\cdot\frac{1}{y} = \frac{2}{t^2}$ $\quad v = \frac{1}{y},\; v' = \frac{-1}{y^2}y'$

$\rightsquigarrow \; -v' + \frac{1}{t}v = \frac{2}{t^2}$

$\rightsquigarrow \; v' - \frac{1}{t}v = \frac{-2}{t^2}$

$$\boxed{\begin{aligned}\mu(t) &= \exp\left(\int -\tfrac{1}{t}\,dt\right)\\ &= \exp(-\ln(t))\\ &= \exp(\ln(1/t))\\ &= 1/t\end{aligned}}$$

$\int \frac{d}{dt}\left[\frac{1}{t}v\right]dt = \int -\frac{2}{t^3}\,dt$

$\frac{1}{t}v = \frac{1}{t^2} + C$

$v = \frac{1}{t} + Ct \; \rightsquigarrow \; \frac{1}{y} = \frac{1}{t} + Ct$

$\frac{1}{2} = 1 + C \; \rightsquigarrow \; C = -\frac{1}{2}$

$\rightsquigarrow \; \frac{1}{y} = \frac{2 - t^2}{2t}$

(B) Exact $\cancel{\partial}\;\cancel{M=}$

$\cancel{2x^3}$ $M = -3x^2 + 4y^2, \; N = 8xy + 8y^2$

$M = -x^3 + 4xy^2 + h(y), \; \frac{\partial}{\partial y}M = 8xy + h'(y) = 8xy + 8y^2$

$h(y) = \frac{8}{3}y^3 + C \qquad x = 0, y = 1$

$\psi(x,y) = -x^3 + 4xy^2 + \frac{8}{3}y^3 = \frac{8}{3}$