SECTION: ____________

Midterm 2

MA266, Sinclair, Fall 2015

November 10, 2015

NAME: Solutions

I have read and understood the Student Honor Code, and this exam reflects my unwavering commitment to the principles of academic integrity and honesty expressed therein.

SCORES:

1. ____________ /10

2. ____________ /10

3. ____________ /10

4. ____________ /10

5. ____________ /10

Total: ____________ /50

TO RECEIVE FULL CREDIT, PLEASE SHOW ALL WORK.

Problem 1 (10 points total) *Circle all correct responses.*

(A) For which r *is* $y = t^r$ *a solution to* $t^2y'' + 4ty' + 3y = 0$*?*

(a) $r = (-3 + i\sqrt{3})/2$

(b) $r = 1$

(c) $r = (-1 + i)/2$

(d) $r = -3$

(e) *none.*

$y' = rt^{r-1}$

$y'' = r(r-1)t^{r-2}$

$(r(r-1) + 4r + 3)t^r = 0$

(B) What is the best guess for a solution to $y'' + 9y = t\sin(3t) + e^{-t}\sin(3t)$*?*

(a) $Y = (At + B)\sin(3t) + Ce^{-t}\sin(3t)$

(b) $Y = (At+B)\sin(3t) + (Ct+D)\cos(3t) + Ee^{-t}\sin(3t) + Fe^{-t}\cos(3t)$

(c) $Y = (At^2+Bt)\sin(3t) + (Ct^2+Dt)\cos(3t) + Ee^{-t}\sin(3t) + Fe^{-t}\cos(3t)$

(d) $Y = (At^2 + Bt)\sin(3t) + (Ct^2 + Dt)\cos(3t) + Ete^{-t}\sin(3t) + Fte^{-t}\cos(3t)$

(e) $Y = (At+B)\sin(3t) + (Ct+D)\cos(3t) + Ete^{-t}\sin(3t) + Fte^{-t}\cos(3t)$.

(C) On which of the following intervals does a solution of

$$(t + 2)y^{(4)} - 3(t + 1)y''' + 3\ln(|t|)y' - 7(t - 3)y = \sec(t)^2$$

exist? (a) $(-1, 3)$ (b) $(-1, \pi/2)$ (c) $(-\pi/2, 0)$ (d) $(0, 1)$ (e) $(-2, 0)$.

(B) $\sin(3t)$ is a solution of $y'' + 9y = 0$
("root" $3i$)

$e^{-t}\sin(3t)$ is not a solution ($-1+3i$ not a root of r^2+9)

(c) domain $t \neq -2, 0, -\pi/2, \pi/2, \ldots$

Problem 2 (10 points total) *Solve the following. (5 points each.)*

(A) $y'' - 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = -2$.

(B) $t^2y'' - 3ty' + 4y = 0$, $y_1(t) = t^2$. *Find the general solution.*

(A) characteristic equation $r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$y_1 = e^{2t}$, $y_2 = te^{2t}$, $y = C_1e^{2t} + C_2te^{2t}$

$y(0) = 1 = C_1 e^{2\cdot 0} + C_2 \cdot 0 \cdot e^{2\cdot 0} \rightsquigarrow C_1 = 1$

$y'(0) = -2 = 2e^{2\cdot 0} + 2C_2 \cdot 0 \cdot e^{2\cdot 0} + C_2 e^{2\cdot 0} \rightsquigarrow C_2 = -4$

(B) Reduction of Order

$y_2 = v \cdot t^2$, $y_2' = v't^2 + 2vt$,

$y_2'' = v''t^2 + 2v't + 2v't + 2v$

$v''t^4 + 4v't^3 + \cancel{2vt^2} - 3v't^3 - \cancel{6vt^2} + \cancel{4vt^2} = 0$

$\rightsquigarrow t^4v'' + t^3v' = 0$ $w = v'$

$w' + \frac{1}{t}w = 0$ linear 1st order homogeneous

$\therefore$ solution is $y = \frac{1}{\mu} = \exp\left(-\int \frac{1}{t}\,dt\right)$

$= \exp(-\ln(t)) = \frac{1}{t} \longrightarrow v = \int \frac{1}{t}\,dt = \ln(t)$

$y = C_1t^2 + C_2\ln(t)t^2$

Problem 3 (10 points total.) *A spring is stretched 6 in by a mass weighing 16 lbs. The mass is attached to a dashpot mechanism that has a damping constant of 2 lb.s/ft. The system is acted on by an external force of* cos(4t) *lbs. If the system is initially at equilibrium find the solution.*

$16 = mg = k \cdot \frac{1}{2}\text{ ft} \rightsquigarrow k = 32$

$g = 32, \quad m = \frac{1}{2}, \quad \gamma = 2$

$\frac{1}{2}u'' + 2u' + 32 = \cos(4t)$

or $u'' + 4u' + 64 = 2\cos(4t)$

$u(0) = 0, \quad u'(0) = 0$

roots of homogeneous part

$r = \frac{-4 \pm \sqrt{16 - 4 \cdot (64)}}{2}$

$= \frac{-4 \pm 4\sqrt{1-16}}{2}$

$= -2 \pm 2\sqrt{15}\, i$

$Y = A\cos(4t) + B\sin(4t)$

$Y' = \cancel{4A\cos} \; -4A\sin(4t) + 4B\cos(4t)$

$Y'' = -16A\cos(4t) \cancel{+} - 16B\sin(4t)$

	$\cos(4t)$	$\sin(4t)$
Y''	$-16A$	$-16B$
$4Y'$	$16B$	$-16A$
$64Y$	$64A$	$64B$
	2	0

$\cancel{48A}$

$\cancel{48A}\; 48A + 16B = 2$

$-16A + 48B = 0$

$3A + B = 1/8$

$-A + 3B = 0$

$\rightsquigarrow B = 1/80, \; A = 3/80$

$u = C_1 e^{-2t}\cos(2\sqrt{15}\,t) + C_2 e^{-2t}\sin(2\sqrt{15}\,t) + \frac{3}{80}\cos(4t) + \frac{1}{80}\sin(4t)$

$0 = u(0) = C_1 + 3/80 \rightsquigarrow C_1 = -3/80$

$0 = u'(0) = \cancel{\frac{3}{80}} \cancel{+} \frac{6}{80} + 2\sqrt{15}\, C_2 + 0 + \frac{4}{80}$

Problem 4 (10 points total) *Find the general solution of the following differential equations. (5 points each.)*

(A) $t^2y'' - 2y = 2t^2 - 3t$, $y_1(t) = t^{-1}$, $y_2(t) = t^2$.

(B) $y''' - 3y'' + 2y' = \sin(t) - 2\cos(t)$.

(A) Variation of Parameters. Step 1 – Standard form

$$y'' - \frac{2}{t^2}y = 2 - \frac{3}{t}, \quad W = \begin{vmatrix} \frac{1}{t} & -\frac{1}{t^2} \\ t^2 & 2t \end{vmatrix} = 2+1=3$$

$$Y = -\frac{1}{t}\int_{t_0}^{t}\frac{(2s^2-3s)ds}{3} + t^2\int_{t_0}^{t}\frac{\left(\frac{2}{s}-\frac{3}{s^2}\right)}{3}ds$$

$$= -\frac{1}{t}\left[\frac{2}{9}s^3 - \frac{1}{2}s^2\right]\Big|_{t_0}^{t} + t^2\left[\frac{2}{3}\ln(|s|) + \frac{1}{s}\right]\Big|_{t_0}^{t}$$

$$Y_1 = -\frac{2}{9}t^2 + \frac{1}{2}t + \frac{2}{3}\ln(t)t^2 + t$$
$$+ \frac{2}{9}t_0^3\frac{1}{t} - \frac{1}{2}t_0^2\frac{1}{t} - \frac{2}{3}\ln(t_0)t^2 - \frac{1}{t_0}t^2$$

$t_0 > 0$

(B) $w = y'$ $\quad w'' - 3w' + 2w = \sin(t) - 2\cos(t)$

characteristic equation B $\quad r^2 - 3r + 2 = (r-1)(r-2) = 0$

$$Y = A\cos(t) + B\sin(t)$$
$$Y' = -A\sin(t) + B\cos(t)$$
$$Y'' = -A\cos(t) - B\sin(t)$$

	$\cos(t)$	$\sin(t)$
Y''	$-A$	$-B$
$-3Y'$	$-3B$	$+3A$
$2Y$	$2A$	$2B$
	-2	1

$A - 3B = -2$

$3A + B = 1$

$10A = 1 \rightsquigarrow A = \frac{1}{10}, \; B = \frac{7}{10}$

$$W = C_1e^t + C_2e^{2t} + \frac{1}{10}\cos(t) + \frac{7}{10}\sin(t)$$

$$\to y = C_1e^t + C_2e^{2t} + C_3 + \frac{1}{10}\sin(t) - \frac{7}{10}\cos(t)$$

Problem 5 (10 points total) *Solve the following problems. (5 points each.)*

(A) $y'' + y' - 2y = te^t$, $y(0) = 1$, $y'(0) = 0$.

(B) $y^{(4)} - 2y'' + y = \cosh(t)$. *Write the correct form of a solution. You DO NOT need to determine the constants.*

(A) Characteristic $(r^2 + r - 2) = (r+2)(r-1) = 0$

$\rightarrow$ e^t a solution.

guess $Y = t(At + B)e^t$

$Y' = (2At + B)e^t + (At^2 + Bt)e^t$

$Y'' = \cancel{2A+B}\ 2Ae^t + (2At+B)e^t$
$+ (2At + B)e^t + (At^2 + Bt)e^t$

	t^2e^t	te^t	e^t
Y''	A	$4A+B$	$2A+2B$
Y'	A	$2A+B$	B
$-2Y$	$-2A$	$-2B$	0
	0	1	0

$6A - 0B = 1$
$2A + 3B = 0$
$\cancel{A = 1/8,\ B = 1/4}$
$A = 1/6, \ B = -1/9$

$y = C_1 e^t + C_2 e^{2t} + \frac{1}{6} t^2 e^t - \frac{1}{9} te^t$

$1 = y(0) = C_1 + C_2$ $\qquad C_2 = -8/9$

$0 = y'(0) = C_1 + 2C_2 - \frac{1}{9}$ $\qquad C_1 = 17/9$

(B) $r^4 - 2r^2 + 1 = (r^2 - 1)^2 = 0$ $\qquad$ cosh, sinh both solutions to $y'' - y = 0 \leftrightarrow r^2 - 1 = 0$

$Y = C_1 e^t + C_2 te^t + C_3 e^{-t} + C_4 te^{-t} + At^2 \cosh(t)$
$+ Bt^2 \sinh(t)$