Exam 2

Work 4 problems for 5 points each.

Problem 0. Define what it means for a sequence (x_n) to be Cauchy. For a bounded sequence (y_n), define $\limsup_{n\to\infty} y_n$. Give a definition of continuity for a function $f : E \to \mathbb{R}$.

A sequence is Cauchy if $\forall \epsilon > 0 \exists N$ such that $|x_m - x_n| < \epsilon$ for all $m, n \geq N$.

$\limsup_{n\to\infty} y_n$ is the supremum of all limits of all convergent subsequences of (y_n).

A function $f : E \to \mathbb{R}$ is continuous if for every $x \in E$ for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$ and $y \in E$.

Problem 1. Prove or give a counterexample to the following. The closure of a set E is the smallest closed set containing E. Every nonempty closed set contains an isolated point or a closed interval.

Let F be a closed set containing E. Any limit point of E is also a limit point of F so $E' \subset F$, thus $\overline{E} \subset F$.

The Cantor ternary set is closed, but contains no isolated points or intervals.

Problem 2. Let $\{[a_i, b_i] : i \in \mathbb{N}\}$ be a sequence of disjoint, closed intervals with $|a_i - b_i| \geq 1$. Show that $\bigcup_{i=1}^{\infty} [a_i, b_i]$ is closed.

Let x be a limit point. For any $\epsilon \in (0, 1/2)$, since $|a_i - b_i| \geq 1$, the set $(x-\epsilon, x+\epsilon)$ can intersect at most two of the intervals. Thus x is a limit point of the union of those two intervals and so must belong to one or the other since the union of two closed sets in closed.

Problem 3. Suppose $f : \mathbb{R} \to (0, \infty)$ is continuous and $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$. Show that f has a maximum value.

For any $N \in \mathbb{N}$, the function f has a (positive) maximum m_N on $[-N, N]$ since this set is closed and bounded. The maxima form an increasing sequence (m_N). This sequence cannot be strictly increasing or else there is a unbounded sequence of points where the function increases away from zero, contradicting that $\lim_{x\to\pm\infty} f(x) = 0$. Thus the sequence (m_N) is eventually constant, so the limit is the maximum value of the function.

Problem 4. What property of an interval is used to prove the Intermediate Value Theorem? A function $f : (a, b) \to \mathbb{R}$ is said to be locally constant if for every $x \in (a, b)$ there is some open interval containing x on which the function is constant. Show that every locally constant function on (a, b) is constant.

The connectedness of an interval is used to prove IVT.

If f is locally constant then if $f(x) = c$ there is a $\delta > 0$ such that $f(y) = c$ for all $y \in (x-\delta, x+\delta)$. So for any $c \in \mathbb{R}$ the sets $O_1 := \{x \in (a, b) : f(x) = c\}$ and $O_2 := \{x \in (a, b) : f(x) \neq c\}$ are both open since all points are interior. Since f is a function, they are also disjoint and their union is

(a, b). If f is not constant, then for c in the range of f, both sets are not empty, contradicting connectedness of (a, b).

Problem 5. Let $f : [a, b] \to \mathbb{R}$ be a function so that for any (x_n) Cauchy sequence in $[a, b]$, the sequence $(f(x_n))$ is still Cauchy. Show that f is uniformly continuous.

Given $x_0 \in [a, b]$, let (x_n) be a sequence in $[a, b]$ so that $x_n \to x_0$. The sequence $x_1, x_0, x_2, x_0, x_3, x_0, \ldots$ is Cauchy as it converges. Thus $f(x_1), f(x_0), f(x_2), f(x_0), f(x_3), f(x_0), \ldots$ is still Cauchy so it must converge to $f(x_0)$. Thus $\lim_{n\to\infty} f(x_n) = f(x_0)$, so f is continuous by the sequential version of continuity. We know any continuous function on a closed bounded set is uniformly continuous.