MA 34100 Fall 2016, HW 1

September 6, 2016

1 $A\ 2.1 \cdots\cdots\cdots\cdots (3.5\ pts)$

a)[0.5 pts] Show that $A \cup B = B$ if and only if $A \subset B$.
($\Rightarrow$) $\forall x \in A$, since $A \subset A \cup B$,and $A \cup B = B$, then $x \in B$, thus $A \subset B$.
($\Leftarrow$) since $A \subset B$, thus $A \cup B \subset B \cup B = B$. Combined with $B \subset A \cup B$, can get $A \cup B = B$.

b)[0.5 pts] Show that $A \cap B = A$ if and only if $A \subset B$.
($\Rightarrow$) $\forall x \in A$, since $A = A \cap B$, $x \in A \cap B$. since $A \cap B \subset B$, thus $x \in B$, which means $A \subset B$.
($\Leftarrow$) since $A \subset B$, thus $A = A \cap A \subset A \cap B$. Combined with $A \cap B \subset A$, can get $A \cap B = B$.

c)[0.5 pts] Show that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
1),$(A \cup B) \cap C \supset (A \cap C) \cup (B \cap C) : A \cap C \subset (A \cup B) \cap C$ and $B \cap C \subset (A \cup B) \cap C$, thus $(A \cup B) \cap C \supset (A \cap C) \cup (B \cap C)$.
2),$(A \cup B) \cap C \subset (A \cap C) \cup (B \cap C) : \forall x \in (A \cup B) \cap C$ means $x \in C\ and\ x \in A\ or\ B$. if $x \in A, then\ x \in A \cap C, then\ x \in (A \cap C) \cup (B \cap C), if\ x \in B, then\ x \in B \cap C, then\ x \in (A \cap C) \cup (B \cap C), thus\ (A \cup B) \cap C \subset (A \cap C) \cup (B \cap C)$.

d)[0.5 pts] Show that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.
from (c) we can get: $(A \cup C) \cap (B \cup C) = (A \cap (B \cup C)) \cup (C \cap (B \cup C)) = ((A \cap B) \cup (A \cap C)) \cup ((C \cap B) \cup (C \cap C)) = (A \cap B) \cup (A \cap C) \cup (B \cap C) \cup (C \cap C) = (A \cap B) \cup C$.

e)[0.5 pts] Show that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.
by definition $(A \cup B) \setminus C = (A \cup B) \cap C^c$. use (c), we can get $(A \cup B) \setminus C = (A \cup B) \cap C^c = (A \cap C^c) \cup (B \cap C^c) = (A \setminus C) \cup (B \setminus C)$.

f)[0.5 pts] Show that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.
$(A \setminus C) \cap (B \setminus C) = (A \cap C^c) \cap (B \cap C^c) = A \cap C^c \cap B \cap C^c = A \cap B \cap C^c = (A \cap B) \setminus C$.

g)[0.5 pts] Show that $\{x \in R : x^2 + x < 0\} = (-1, 0)$.
$\{x \in R : x^2 + x < 0\} = \{x \in R : x > 0\ and\ x + 1 < 0\} \cup \{x \in R : x < 0\ and\ x + 1 > 0\}$. first set is empty, second set equals to (-1,0). Then $\{x \in R : x^2 + x < 0\} = (-1, 0)$.

2 $A\ 2.3 \cdots\cdots\cdots (1.5\ pts)$

a)[0.5 pts] Describe $\cup_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$ and $\cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$.
$\cup_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = (-1, 1)$.
$\cap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$.

b)[0.5 pts] Describe $\cup_{n=1}^{\infty}(-n,n)$ and $\cap_{n=1}^{\infty}(-n,n)$.
$\cup_{n=1}^{\infty}(-n,n) = (-\infty,\infty)$.
$\cap_{n=1}^{\infty}(-n,n) = (-1,1)$.

c)[0.5 pts] Describe $\cup_{n=1}^{\infty}[n,n+1]$ and $\cap_{n=1}^{\infty}[n,n+1]$.
$\cup_{n=1}^{\infty}[n,n+1] = [1,\infty)$.
$\cap_{n=1}^{\infty}[n,n+1] = \emptyset$.

3 $A\ 2.6 \cdots\cdots\cdots (2\ pts)$

a), Show that $f : R \to R$ is one-to-one if and only if $f(A \cap B) = f(A) \cap f(B)$. for all sets A, B.
($\Rightarrow$), 1),since $f(A \cap B) \subset f(A), f(B)$, thus $f(A \cap B) \subset f(A) \cap f(B)$.
2),$\forall y \in f(A) \cap f(B)$, means $\exists a \in A, b \in B$ s.t. $f(a) = f(b) = y$, since f is one-to-one, thus $a = b \in A \cap B$, then $y \in f(A \cap B)$. Thus, if f is one-to-one, then $f(A \cap B) = f(A) \cap f(B)$.
($\Leftarrow$), if f is not one-to-one, then $\exists a, b, y,\ s.t.\ a \neq b\ and\ f(a) = f(b) = y$. Let $A = \{a\}, B = \{b\}, A \cap B = \emptyset$, thus $f(A \cap B) = \emptyset$. However, $f(A) = f(B) = \{y\}$.Thus, it is contradictive with $f(A \cap B) = f(A) \cap f(B)$.

b), Show that $f : R \to R$ is one-to-one if and only if $f(A) \cap f(B) = \emptyset$ for all sets A, B with $A \cap B = \emptyset$.
($\Rightarrow$), if $A \cap B = \emptyset$,then$f(A \cap B) = \emptyset$, from (a) we know, if f is one-to-one,then $\emptyset = f(A \cap B) = f(A) \cap f(B)$.
($\Leftarrow$), if f is not one-to-one, then $\exists a, b, y,\ s.t.\ a \neq b\ and\ f(a) = f(b) = y$. Let $A = \{a\}, B = \{b\}, A \cap B = \emptyset$, thus $f(A \cap B) = \emptyset$. However, $f(A) = f(B) = \{y\} \neq \emptyset$. It is contradictive with the assumption $f(A) \cap f(B) = \emptyset$ for all sets A, B with $A \cap B = \emptyset$.

4 $A\ 2.7 \cdots\cdots\cdots (3\ pts)$

This exercise concerns the notion of preimage. If $f : X \to Y$and $E \subset Y$, then

$$f^{-1}(E) = \{x : f(x) = y\ for\ some\ y \in E\} \subset X$$

is called the preimage of E under f. [There may or may not be an inverse function here; $f^{-1}(E)$ has a meaning even if there is no inverse function.]

a)[0.5 pts] Show that $f(f^{-1}E) \subset E$ for every set $E \subset R$.
$\forall y \in f(f^{-1}(E))$, means $\exists x \in f^{-1}(E)$ s.t.$f(x) = y$, since $x \in f^{-1}(E)$, by definition of $f^{-1}(E)$, means that $y = f(x) \in E$. thus $f(f^{-1}(E)) \subset E$ for every set $E \subset R$.

b)[0.5 pts] Show that $f^{-1}(f(E)) \supset E$ for every set $E \subset R$.
$\forall x \in E$,let $y = f(x) \in f(E)$, since $y \in f(E)$, by definition $x \in f^{-1}(f(E))$, thus $f^{-1}(f(E)) \supset E$ for every set $E \subset R$.

c)[0.5 pts] Can you simplify $f^{-1}(A \cup B)$ and $f^{-1}(A \cap B)$?
$f^{-1}(A \cup B)$ can be simplified as $f^{-1}(A) \cup f^{-1}(B)$. $f^{-1}(A \cap B)$ can not be simplified.

d)[0.5 pts] Show that $f : R \to R$ is one-to-one if and only if $f^{-1}(\{b\})$ contains at most a single point for any $b \in R$.

($\Rightarrow$) : if $\exists b \in R$,s.t.$f^{-1}(\{b\})$ contains at least two points x_1, x_2. Then by definition $f(x_1) = f(x_2) = b$ which is contradictive with assumption f is one-to-one.

($\Leftarrow$) : if f is not one-to-one, then $\exists x_1, x_2, y$, s.t. $f(x_1) = f(x_2) = y$. Then it means $x_1, x_2 \in f^{-1}(\{y\})$ which is contradictive with assumption $f^{-1}(\{b\})$ contains at most a single point for any $b \in R$.

e)[0.5 pts] Show that $f : R \to R$ onto, that is, the range of f is all of R if and only if $f(f^{-1}(E)) = E$ for every set $E \subset R$.

($\Rightarrow$) : $\forall y \in E$, since f is onto, thus $\exists x \in R$,s.t. $f(x) = y$, thus by definition $x \in f^{-1}(E)$, since $y = f(x) \in f(f^{-1}(E))$, thus $f(f^{-1}(E)) \supset E$. using (a) $f(f^{-1}(E)) \subset E$, get $f(f^{-1}(E)) = E$.

($\Leftarrow$) : if f is not onto, then $\exists y \in R$, s.t. $\forall x \in R, f(x) \neq y$. Then let $E = \{y\}, f^{-1}(E) = \emptyset, f(f^{-1}(E)) = \emptyset \neq E = \{y\}$. Which is contradictive with assumption $f(f^{-1}(E)) = E$ for every set $E \subset R$.