MATH 351, FALL 2017, EXAM #1

Instructions: Work the first problem and then exactly three of the remaining four problems. In order to receive full credit be sure to show all work. Each problem is worth 5 points.

Problem 1. Short answer.

2 pts (A) Give the precise definition for when a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ in a vector space V is linearly independent.

2 (B) Give one possible definition of the rank of a system of linear equations.

1 (C) If $T : V \to W$ is a linear transformation show that $T(\vec{0}) = \vec{0}$.

Respond to three of the problems below.

Problem 2. Consider the system of linear equations:

$$\left\{ \begin{array}{c} x - y + 2z + u = 1 \\ 2x + y + 2u + 3w = 4 \\ x + z + u + w = 3 \end{array} \right\}$$

1 (A) Express the system as an augmented matrix.

3 (B) Put the augmented matrix into row reduced echelon form.

1 (C) Express the solution set in parametric form.

Problem 3. Let $C(\mathbb{R})$ be the vector space of all continuous, real-valued functions with domain $\mathbb{R}$.

3 (A) Show that the set $C^2(\mathbb{R})$ of all functions whose first and second derivatives exist everywhere is a subspace of $C(\mathbb{R})$.

2 (B) Give three examples of functions f such that for each one $\{f, f', f''\}$ is a linearly dependent set. The functions you pick are not allowed to be linear combinations of each other.

Problem 4. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -1 \\ 6 & 2 & 0 \end{bmatrix}$$

3 (A) Find the null space $\mathrm{Nul}(A)$.

2 (B) Find a non-zero vector $\vec{b}$ such that $A \cdot \vec{x} = \vec{b}$ is consistent.

5 **Problem 5.** Let $A \in M_{kn}(\mathbb{R})$ be a $k \times n$ real matrix. If $k > n$ show that $\mathrm{Col}(A) \neq \mathbb{R}^k$. Give as precise of a logical argument as you can.

1.A. $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent if no vector ~~in~~ the set can be expressed as a linear combination of the other vectors.

1.B. The rank is the smallest number of linear equations for a linear system with the same solution set

1.C. $T(\vec{0}) = T(0 \cdot \vec{0}) = 0 \cdot T(\vec{0}) = \vec{0}$.

2.A.

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 & 1 & 3 \end{array}\right]$$

2.B. RREF is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 6 \\ 0 & 0 & 1 & 0 & 0 & 4 \end{array}\right]$$

2.C. parametric form of solution is

$$\begin{bmatrix} -1 \\ 6 \\ 4 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

3.A. $f \in C^2(\mathbb{R})$, then f' exists everywhere,
so f is continuous, i.e., $f \in C(\mathbb{R})$

$(f+g)' = f' + g'$, $(f+g)'' = f'' + g''$, ~~...~~
if f', f'', g', g'' exist.

Also $(c \cdot f)' = c \cdot f'$, $(cf)'' = c \cdot f''$.

Therefore $C^2(\mathbb{R})$ is a subspace of $C(\mathbb{R})$.

3.B. $1, x, e^x, e^{2x}, \sin(x), \sin(2x)$,
~~cos(2~~ $\cos(x)$, $\cos(2x)$, ...

4.A.

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & -2 & -1 & 0 \\ 6 & 2 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -8 & -3 & 0 \\ 0 & -16 & -6 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 8 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/8 & 0 \\ 0 & 1 & 3/8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$Nul(A) = x_3 \cdot \begin{bmatrix} 1/8 \\ -3/8 \\ 1 \end{bmatrix}$$

4.B. $A \cdot \vec{x} = \vec{b}$ is consistent exactly when $\vec{b} \in \mathrm{Col}(A)$

$$\mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

so any of these vectors will work.

In general $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathrm{Col}(A)$ exactly when

$$2(x+y) = z.$$

5. A has more rows than columns, ~~to~~ so the RREF will have a zero row. Thus

$$\left[\begin{array}{c|c} & 0 \\ * & 0 \\ & \vdots \\ & 0 \\ 0 \; 0 \ldots 0 & 1 \end{array}\right]$$ is inconsistent

Undoing row operations we see that

$[A \mid \vec{b}]$ is inconsistent for some $\vec{b} \in \mathbb{R}^k$, so $\vec{b} \notin \mathrm{Col}(A)$.