MATH 351, FALL 2017, EXAM #2

Instructions: Work the first problem and then exactly three of the remaining four problems. In order to receive full credit be sure to show all work. Each problem is worth 5 points.

Problem 1. Short answer.

2 (A) Give the precise definition of a vector space having dimension n.

2 (B) Define the rank of a matrix and give a full statement of the Rank-Nullity Theorem.

1 (C) Explain why a maximal linearly independent set of vectors in a vector space V is a basis for V

Respond to three of the problems below.

Problem 2. Consider the matrix:

$$\begin{bmatrix} 1 & 2 & -2 & -3 & 1 \\ 0 & -2 & 0 & 4 & -2 \\ 2 & 1 & -4 & 3 & 4 \\ 1 & 0 & -2 & 1 & -1 \end{bmatrix}$$

2 (A) Find a basis for the row space.

2 (B) Find a basis for the column space.

1 (C) Extend the basis of the row space to a basis for $\mathbb{R}^5$.

Problem 3. Consider the matrices:

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

2 (A) Find all eigenvalues of the matrices A and B

2 (B) For each eigenvalue of A and B describe the set of eigenvectors.

1 (C) Find an eigenvector for A^{-1} and an eigenvector for B^{-1}.

Problem 4. Let A and B be real matrices.

2.5 (A) If $\{A\vec{v}_1, \ldots, A\vec{v}_n\}$ is a basis for Col(A), find all possible dimensions of the subspace spanned by $\vec{v}_1, \ldots, \vec{v}_n$. Be sure to explain.

2.5 (B) If B is a square matrix and $B^{2017} = 0$, find all possible eigenvalues. Be sure to explain.

5 **Problem 5.** Suppose that A and B are $n \times n$ real matrices. If A^2B is invertible, show that A and B are both invertible. Give as precise of a logical argument as you can.

1.A. The dimension of a vector space is the least number of elements in a spanning set (basis). So a vector space ~~f~~ has dimension n if the least number of elements in a spanning set is n.

1.B. The rank of a matrix is the dimension of its row space.

Rank-Nullity states that for a matrix $A \in M_{kn}(\mathbb{R})$

$$\dim \operatorname{Row}(A) + \dim \operatorname{Nul}(A) = n$$

1.C. If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a maximal linearly independent set, then for every $\vec{w} \in V$ $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{w}\}$ is linearly dependent, which implies that $\vec{w} \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ since there is no dependency among the $\vec{v}_i$'s.
Therefore $\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\} = V$.

2.A.

$$\begin{bmatrix} 1 & 2 & -2 & -3 & 1 \\ 0 & -2 & 0 & 4 & -2 \\ 2 & 1 & -4 & 3 & 4 \\ 1 & 0 & -2 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -2 & -3 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

echelon form

Rows 1, 2, and 3 contain the pivots so

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 5 \end{bmatrix} \right\}$$

span Row(A) and are linearly independent.

2.B. Columns 1, 2, and 4 contain a pivot, so

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 3 \\ 1 \end{bmatrix} \right\}$$ form a basis for $Col(A)$.

2.C. Columns 3 and 5 are missing a pivot, so adding $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ to ~~a~~ the basis of $Row(A)$

gives a set of 5 vectors spanning $\mathbb{R}^5$ which makes this a basis as $\dim \mathbb{R}^5 = 5$.

3.A. $\det(A - \lambda I) = (1-\lambda)(5-\lambda) + 6 = \lambda^2 - 6\lambda + 11$

$$\lambda = \frac{6 \pm \sqrt{36 - 4 \cdot 1 \cdot 11}}{2} = 3 \pm i\sqrt{2}$$

$\det(B - \lambda I) = (3-\lambda)(4-\lambda) - 2 = \lambda^2 - 7\lambda + 10$
$= (\lambda - 2)(\lambda - 5)$

$\lambda = 2, 5$

3.B. $\lambda = 3 + i\sqrt{2}$

$$\left[\begin{array}{cc|c} -2 - i\sqrt{2} & 3 & 0 \\ -2 & 2 - i\sqrt{2} & 0 \end{array}\right]$$ > dependent rows by $\frac{2 - i\sqrt{2}}{3}$

$\cancel{R_2}$ $$\left[\begin{array}{cc|c} 1 & -1 + i\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

eigenvectors $c \cdot \begin{bmatrix} 1 - i\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$ $c \neq 0$

$x_2 = 1, \; x_1 = 1 - i\frac{\sqrt{2}}{2}$

For $\lambda = 3 - i\sqrt{2}$ eigenvector is conjugate: $c \cdot \begin{bmatrix} 1 + i\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$

3.B. (cont'd) $\lambda = 2$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right]$$

$x_2 = 1, \; x_1 = -2$ eigenvectors $c \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$c \neq 0$

$\lambda = 5$ $$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

eigenvectors $c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_2 = 1, \; x_1 = 1$

$c \neq 0.$

3.C. $A\vec{x} = \lambda \vec{x} \Rightarrow \vec{x} = A^{-1}(\lambda \vec{x}) \Rightarrow \vec{x} = \lambda (A^{-1}\vec{x})$

$\Rightarrow A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x} \quad \lambda \neq 0$

so $\begin{bmatrix} 1 - i\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$ is an eigenvector for A^{-1}

and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for B^{-1}.

4.A. If $a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n = \vec{0}$

then $A(a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n) = A\vec{0}$

$\parallel$

$a_1(A\vec{v}_1) + a_2(A\vec{v}_2) + \dots + \cancel{A}\, a_n(A\vec{v}_n) = \vec{0}$

Thus $a_1 = a_2 = \dots = a_n = 0$ since $A\vec{v}_i$'s are linearly independent

~~th~~ So $\{\vec{v}_1, \dots, \vec{v}_n\}$ are linearly independent and

~~s~~ $\dim \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\} = n$.

4.B. If $B\vec{x} = \lambda\vec{x}$, then $B^{2017}\vec{x} = \lambda^{2017}\vec{x}$

$B^{2017}\cancel{x} = 0$ shows that $\lambda^{2017}\vec{x} = \vec{0}$

which implies that $\lambda = 0$ since $\vec{x} \neq \vec{0}$.

Therefore $\lambda = 0$ is the only possible eigenvalue for B.

5. A^2B ~~is~~ is invertible

If $B\vec{x} = \vec{0}$ then $A^2B\vec{x} = A^2\vec{0} = \vec{0}$, which implies that

$\vec{x} = \vec{0}$ since A^2B invertible

Thus $\operatorname{rank} B = n - \operatorname{nullity} B = n - 0 = n$ so B

is invertible.

$A^2 = A^2BB^{-1}$ invertible so apply same reasoning to

see that AB invertible.