MATH 351, FALL 2017, EXAM #3

Instructions: Work the first problem and then exactly three of the remaining four problems. In order to receive full credit be sure to show all work. Each problem is worth 5 points.

Problem 1. Short answer.

2 (A) Define what it means for $\{\vec{v}_1, \ldots, \vec{v}_n\}$ to be an orthonormal basis for $\mathbb{R}^n$.

2 (B) State the Spectral Theorem for a real symmetric matrix A.

1 (C) Give an example of a 3×3 real matrix which is not diagonalizable.

Respond to three of the problems below.

Problem 2. Diagonalize the following matrices, if possible.

3 (A) $A = \begin{bmatrix} 1 & 8 & -3 \\ 0 & -1 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

2 (B) $B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, given that $(-1,1,0)^t$ and $(1,1,2)^t$ are eigenvectors.

Problem 3. Solve the following.

2 (A) Find the inverse of the matrix

$$A = \begin{bmatrix} 7 & 0 & -7 & 0 \\ 1 & 2 & 1 & 2 \\ 2 & -1 & 2 & -1 \\ 0 & -7 & 0 & 7 \end{bmatrix}.$$

2 (B) Find the least squares solution $\vec{x}$ to the inconsistent system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 2 \end{bmatrix}.$$

1 What is the significance of $A\vec{x}$?

5 **Problem 4.** Write down the matrix of the orthogonal projection from $\mathbb{R}^4$ onto the subspace spanned by $\{(1,-1,2,2), (1,0,-1,0), (1,0,0,-1)\}$.

Problem 5. Let A be a real 3×3 matrix such that $A = A^t$, $A^4 = I$.

2 (A) Give an example of such a matrix where $A \neq I$.

3 (B) Show that $A^2 = I$ for any such matrix.

1.A. $\vec{v}_1, \ldots, \vec{v}_n$ is an orthonormal basis for $\mathbb{R}^n$ if it is a basis and

$$\vec{v}_i \cdot \vec{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

1.B. $A^t = A$ then all eigenvalues are real and there is an orthonormal basis of eigenvectors.

1.C. $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ any non-zero nilpotent matrix will do.

2.A. $\begin{pmatrix} 1 & 8 & -3 \\ 0 & -1 & 4 \\ 0 & 0 & 5 \end{pmatrix}$ is upper triangular so eigenvalues are $\lambda = 1, -1, 5$

$\lambda = 1$ $\begin{pmatrix} 0 & 8 & -3 \\ 0 & -2 & 4 \\ 0 & 0 & 4 \end{pmatrix}$ $x_1 = 1$ free variable, $x_2 = 0$, $x_3 = 0$

$\lambda = -1$ $\begin{pmatrix} 2 & 8 & -3 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{pmatrix}$ $x_2 = 1$ free variable, $x_3 = 0$, $x_1 = -4$

$\lambda = 5$ $\begin{pmatrix} -4 & 8 & -3 \\ 0 & -6 & 4 \\ 0 & 0 & 0 \end{pmatrix}$ $x_3 = 1$ free variable, $x_2 = 2/3$, $x_1 = 7/12$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & -4 & 7/12 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{pmatrix}$$

2.B. $B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is symmetric, so it is diagonalizable

last eigenvector is orthogonal to $(-1,1,0)$ $(1,1,2)$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$x_3 = 1$ free, $x_2 = -1$, $x_1 = -1$

2.B. cont'd. So the last eigenvector is $(-1,-1,1)^t$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \lambda = 1$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} \qquad \lambda = 3$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} \qquad \lambda = 3$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix}$$

3.A. Rows of A are orthogonal so

$$AA^t = \begin{pmatrix} 7 & 0 & -7 & 0 \\ 1 & 2 & 1 & 2 \\ 2 & -1 & 2 & -1 \\ 0 & -7 & 0 & 7 \end{pmatrix}\begin{pmatrix} 7 & 1 & 2 & 0 \\ 0 & 2 & -1 & -7 \\ -7 & 1 & 2 & 0 \\ 0 & 2 & -1 & 7 \end{pmatrix} = \begin{pmatrix} 98 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 98 \end{pmatrix}$$

$$\text{so } A^{-1} = A^t \begin{pmatrix} 1/98 & 0 & 0 & 0 \\ 0 & 1/10 & 0 & 0 \\ 0 & 0 & 1/10 & 0 \\ 0 & 0 & 0 & 1/98 \end{pmatrix}$$

3.B. Least squares solution

$$A^tA\vec{x} = A^t\vec{b} \qquad A^tA = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^t\vec{b} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$\vec{x} = (A^tA)^{-1}A^t\vec{b} = \frac{1}{10}\begin{pmatrix} 2 & -2 \\ -2 & 7 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$A\vec{x}$ is the orthogonal projection of $\vec{b}$ onto the column space of A.

4. $W = \text{span}\{(1\ -1\ 2\ 2)^t, (1\ 0\ -1\ 0)^t, (1\ 0\ 0\ -1)^t\}$

$I = P_W + P_{W^\perp}$

Find $W^\perp$ $\quad \left(\begin{array}{cccc|c} 1 & -1 & 2 & 2 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$

$x_4 = 1$ free

$x_3 = 1$

$x_2 = 5$ $\quad$ so $W^\perp = c\begin{pmatrix} 1 \\ 5 \\ 1 \\ 1 \end{pmatrix}$

$x_1 = 1$

$$P_{W^\perp} = \frac{1}{\|(1\,5\,1\,1)^t\|^2}\begin{pmatrix} 1 \\ 5 \\ 1 \\ 1 \end{pmatrix}(1\,5\,1\,1) = \frac{1}{28}\begin{pmatrix} 1 & 5 & 1 & 1 \\ 5 & 25 & 5 & 5 \\ 1 & 5 & 1 & 1 \\ 1 & 5 & 1 & 1 \end{pmatrix}$$

$$P_W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{28}\begin{pmatrix} 1 & 5 & 1 & 1 \\ 5 & 25 & 5 & 5 \\ 1 & 5 & 1 & 1 \\ 1 & 5 & 1 & 1 \end{pmatrix}$$

5. A. $\quad A = -I \qquad A^4 = I$

5. B. $\quad A^4 = I$ so $\lambda^4 = 1$ for all eigenvalues

By spectral theorem all eigenvalues are real so $\lambda = \pm 1$

Also by spectral theorem $A = Q\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}Q^{-1}$

so $\quad A^2 = Q\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}Q^{-1} = I$