A VERY SHORT PROOF OF THE EXISTENCE OF THE INJECTIVE ENVELOPE OF AN OPERATOR SPACE

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ABSTRACT. We use Ellis' lemma to give a "one-line" proof of the existence of the injective envelope of an operator space first shown by work of Hamana and Ruan.

1. INJECTIVE ENVELOPES

Ramsey theoretic methods provide powerful tools in functional analysis, ergodic theory, and additive combinatorics: see [AT] for many of these applications. The goal of this note is to give a completely elementary and very short proof of the existence of Hamana's injective envelope of a operator space [Ha, Ru] via Ellis' lemma on topological semigroups. We mention that the treatment of injective envelopes as in [Pa1, Pa2, Pa3] is very close in spirit with our proof. In fact, in [Pa3] the injective envelope of certain crossed products of a countable, discrete G is related with idempotents in the Stone+Cech compactification βG, but it seems a formal connection with Ellis' lemma in the general context was never made.

Let $\mathcal{S}$ be a (nonempty) hausdorff topological space equipped with a semigroup operation so that $\mathcal{S} \ni s \mapsto st$ is continuous for each $t \in \mathcal{S}$ separately. We will say that $\mathcal{S}$ is a (left) topological semigroup. Let $\mathfrak{I}(\mathcal{S})$ be the (possibly empty) set of idempotent elements of $\mathcal{S}$ which we equip with the natural partial order $e_1 \prec e_2$ if $e_1e_2 = e_2e_1 = e_1$. A *minimal idempotent* is an idempotent which is a minimal element of the poset $(\mathfrak{I}(\mathcal{S}), \prec)$. We recall an elementary but fundamental result due to Ellis and Furstenberg+Katznelson:

Lemma 1.1 (Ellis, Furstenberg+Katznelson)**.** *If $\mathcal{S}$ is a compact left topological semigroup then $\mathfrak{I}(\mathcal{S})$ is nonempty and for every idempotent e there is a minimal idempotent $f \prec e$.*

This appears as Theorems 1.1 and 1.2 in [FK]: the proof is literally two lines, one for each theorem! (See also [Gl, I.4] for more on this.)

Let $E \subset \mathcal{B}(H)$ be an operator space. Following [Pa2, Chapter 15], we will say that a pair $(\mathcal{E}, \kappa)$ consisting of an operator space $\mathcal{E}$ and a completely isometric embedding $\kappa : E \to \mathcal{E}$ is an *injective envelope* for E if:

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(1) $\mathcal{E}$ is an injective object in the category of operator spaces and completely contractive (c.c.) maps;
(2) $\kappa(E) \subset \mathcal{E}_1 \subset \mathcal{E}$ is injective, then $\mathcal{E}_1 = \mathcal{E}$.

Theorem 1.2 (Hamana, Ruan). *Any operator space has an injective envelope.*

Proof. Let $E \subset \mathcal{B}(H)$ be an operator space. Let $\mathcal{S}$ be the set of all completely contractive maps $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ so that $\varphi|_E = \mathrm{id}_E$. It is clear that $\mathcal{S}$ is a semigroup under composition. Since $E \subset \mathcal{B}(H)$ is closed, it is easy to check that $\mathcal{S}$ is closed in $\mathcal{B}(\mathcal{B}(H), \mathcal{B}(H))$ in the BW-topology, see [Pa2, Chapter 7]; therefore, $\mathcal{S}$ is compact and it is easy to see (say, by [Pa2, Proposition 7.3]) that $\varphi \mapsto \varphi \circ \psi$ is BW-continuous. Therefore, there is map $\varphi \in \mathcal{S}$ such that $\varphi \circ \varphi = \varphi$ and $\varphi \circ \psi = \psi \circ \varphi = \psi$ implies that $\psi = \varphi$ for any other $\psi \in \mathcal{S}$ such that $\psi \circ \psi = \psi$. Since $\mathcal{E} \subset \mathcal{B}(H)$ is injective if and only if it is the image of a c.c. projection $\theta : \mathcal{B}(H) \to \mathcal{E}$, it is trivial to see that $E \hookrightarrow \varphi(\mathcal{B}(H))$ is an injective envelope for E. □

Some further remarks are in order:

I. If $X \subset \mathcal{B}(H)$ is a weakly closed injective subspace, then $\mathcal{B}(X, X)$ inherits the BW-topology is again compact, as is the subsemigroup $\mathcal{CC}(X)$ of completely contractive maps $\phi : X \to X$.

II. If there is a G-action by unital complete isometries on an operator system E and $E \subset X \subset \mathcal{B}(H)$ is weakly closed, injective, and G-invariant, then the G-fixed points of the semigroup $\mathcal{S} \subset \mathcal{CC}(X, X)$ above, $\mathcal{S}^G$, is easily seen to be a BW-closed subsemigroup of $\mathcal{S}$, and the same proof shows the existence of a (relative) G-injective envelope for E: see [KK] for more on this and applications to the theory of reduced group C^*-algebras.

III. If $\pi : G \to \mathcal{U}(H)$ is a unitary representation, then there is a minimal unital completely positive (u.c.p.) π-invariant projecction $E : \mathcal{B}(H) \to \mathcal{B}(H)$. In general there should be many such projections. If π is the left regular representation of G on $\ell^2(G)$, then the image of one such projection lies in $\ell^\infty(G)$, thus corresponds with the Furstenberg+Hamana boundary. If π is amenable in the sense of Bekka [**?**], it is trivial to see that $\mathbb{C}1_{\mathcal{B}(H)}$ is one such subspace.

IV. If X is a dual Banach space, then the contractive operators $\mathcal{C}(X, X)$ is a compact topological semigroup in the BW-topology.

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