It was asked by E. Szemerédi if, for a finite set $A \subset \mathbb{Z}$, one can improve estimates for $\max\{|A+A|, |A \cdot A|\}$, under the constraint that all integers involved have a bounded number of prime factors – that is, each $a \in A$ satisfies $\omega(a) \leq k$. In our paper we show that this maximum is at least of order $|A|^{\frac{5}{3}-o(1)}$ provided $k \leq (\log |A|)^{1-\varepsilon}$ for some $\varepsilon > 0$. In fact, this will follow from an estimate for additive energy which is best possible up to factors of size $|A|^{o(1)}$. Our proof consists of three parts: combinatorial (structural results for sets with small multiplicative doubling), analytical (backwards martingales) and number theoretical (the subspace theorem).

This is joint work of B. Hanson, M. Rudnev, I. Shkredov, and D. Zhelezov.