Let $r_0(n)$ be the number of representations of n as a sum of two squares, and $r_1(n)$ count the number of representations of n as a sum of an integer square and a prime square.

The asymptotic formulas for $r_0^k(n)$, $k \geq 1$ summed over $n \leq x$ are well-known via a classical complex analytic approach. As for $r_1^k(n)$, the only cases with known asymptotics are $k = 0, 1, 2$. For $k = 2$ the main term comes from both trivial solutions (i.e. those with $a^2 + p^2 = b^2 + q^2$ implying $p = q$) and the non-trivial ones. The last ones lie on a very thin set of extremely large values of r_1 and we are interested in getting a better understanding of this set.

We will show that the number of integers $n \leq x$ such that $r_1(n) \geq 1$ equals $\frac{\pi}{2} \cdot \frac{x}{\log x}$ minus a secondary term of size $x/(\log x)^{1+\delta+o(1)}$, where $\delta := 1 - \frac{1+\log\log 2}{\log 2} = 0.0860713320\ldots$ is the Erdős-Tenenbaum-Ford constant.

This, in particular, implies that the main contribution to the sum of $r_1^2(n)$ comes from those integers n with

$$\omega(n) \sim 2\log\log x \text{ and } r_1(n) = (\log x)^{\log 4 - 1 + o(1)}.$$

This is a joint work with Andrew Granville and Cihan Sabuncu.