Rees Algebras of Modules

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Dedicated to David Rees on his eightieth birthday

Abstract

We study Rees algebras of modules within a fairly general framework. We introduce an approach through the notion of Bourbaki ideals that allow the use of deformation theory. One can talk about the (essentially unique) Bourbaki ideal $I(E)$ of a module E which, in many situations, allows to reduce the nature of the Rees algebra of E to that of its Bourbaki ideal $I(E)$. Properties such as Cohen–Macaulayness, normality and being of linear type are viewed from this perspective. The known numerical invariants of an ideal and its associated algebras are considered in the case of modules, such as the analytic spread, the reduction number, the analytic deviation. Corresponding notions of complete intersection, almost complete intersection and equimultiple modules are examined to some detail. A special consideration is given to certain modules which are fairly ubiquitous because interesting vector bundles appear in this way. For these modules one is able to estimate the reduction number and other invariants in terms of the Buchsbaum–Rim multiplicity.

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1 Introduction

The objective of this work is to study the Rees algebra $\mathcal{R}(E)$ of a finitely generated module E over a Noetherian ring R, defined as the symmetric algebra $\mathcal{S}(E)$ of E modulo its R-torsion submodule.

The motivation comes from various sources. One is to prove for Rees algebras of modules many of the recent results that have been obtained in the case of ideals, including estimates on analytic spreads and reduction numbers. As it turns out, this is not just routine generalization, but requires a good deal of technical development. Another motivation comes from the fact that Rees algebras of modules include as a special case the so called multi Rees rings, which correspond to the case where the module is a direct sum of ideals. The importance of the latter is beyond argument as they provide the rings of functions on the blowup of a scheme along several subschemes.

A subtler motivation is the frequency in which symmetric algebras of modules disclose themselves as coordinate rings of certain correspondences in algebraic geometry. Thereafter, the various projections of these varieties may require killing torsion as a preliminary step at the algebra level. Thus, Rees algebras become a natural milieu to envelop such constructions. For example, when E is the module of differentials or the module of differential forms of the homogeneous coordinate ring of a projective k-variety X then $k\otimes\mathcal{R}(E)$ turns out to be the homogeneous coordinate ring of the tangential variety or the Gauss image of X, respectively ([42]).

Some correspondences are defined by ideals generated by bilinear forms in two disjoint sets of variables. For instance, the variety of pairs of commuting matrices is defined up to radical by bilinear forms and it is an open question whether it is ideal theoretically so defined (see [51, Chapter 9] for a discussion). The problem can be rephrased in terms of a suitable module E satisfying $\mathcal{S}(E)=\mathcal{R}(E)$. Modules fulfilling this last condition are said to be of *linear type* and constitute a main source of related questions.

For an arbitrary module one lacks the remarkable interaction that exists for an ideal $I\subset R$ between the Rees algebra and the associated graded ring of I, by means of the canonical surjection $\mathcal{R}(I)\to \mathrm{gr}_R(I)$. This interplay is so crucial in the ideal case that one seeks to remedy its absence by looking at alternative tools. In this work we approach the problem by resorting to the so called Bourbaki ideals ([1]).

Recall that given a module E, an ideal $I\subset R$ is called a *Bourbaki ideal* of E if it fits into an exact sequence $0\to F\longrightarrow E\longrightarrow I\to 0$, where F is a free R-module. The resulting surjection $\mathcal{R}(E)\to \mathcal{R}(I)$ is to play a major role throughout the work. However, to deal efficiently with this idea we first systematically develop a notion of *generic* Bourbaki ideals defined over a suitable faithfully flat extension of R. This approach has of course a classical flavor while carrying the advantage that the Bourbaki ideal $I(E)$ thus obtained is essentially unique. One may wonder when the surjection $\mathcal{R}(E)\to \mathcal{R}(I(E))$ is a deformation (i.e., when the kernel is generated by a regular sequence on $\mathcal{R}(E)$). If this is the case, the map provides a tight interplay between the defining equations of the two algebras and acts as a vessel for transferring properties from $\mathcal{R}(I(E))$ to $\mathcal{R}(E)$ such as Cohen–Macaulayness, normality and being of linear type.

We will now describe the content of this paper.

Section 2 settles a few basic facts around Rees algebras, such as dimension, reductions of modules and analytic spreads. Section 3 contains the bulk of results that shape the theory of generic Bourbaki ideals. The main theorem of the section (Theorem 3.5) states that the Cohen–Macaulayness of the Rees algebra, its normality and the linear type property pass from $I(E)$ to E and vice-versa. Furthermore, if any of these conditions hold then $\mathcal{R}(E)$ is a deformation of $\mathcal{R}(I(E))$. This result is the basic framework of the present work. In proving it we obtain several more general statements that may be of independent interest.

In describing the last two sections, we assume that $(R,\mathfrak{m})$ is a local Cohen–Macaulay ring with infinite residue field and E is a finitely generated R-module with rank $e > 0$, analytic spread ℓ and minimal number of generators n.

In Section 4 we show how the results of the previous section lead to short proofs of known results and also to new structure theorems. To explain some of them, we recall that E is said to satisfy condition G_s, for an integer $s \geq 1$, if $\nu(E_\mathfrak{p}) \leq \dim R_\mathfrak{p} + e - 1$ whenever $1 \leq \dim R_\mathfrak{p} \leq s-1$. Note that such a condition for $s = \dim R$ or $s = \ell - e + 1 \leq \dim R$ affects only the behavior of E on the punctured spectrum (or even less). We show, for example, that if $\mathcal{R}(E)$ is Cohen–Macaulay then the reduction number of E is at most $\ell - e$ (cf. Theorem 4.2). By and large, however, the section is focused on rings of small dimension or modules of small projective dimension. Thus, as a typical result, if $\dim R = 3$ and E is orientable and satisfies G_3, we obtain the Cohen–Macaulayness of $\mathcal{R}(E)$ provided the reduction number of E is at most one. Similar results are described for dimension at most 5. In the situation where E has projective dimension one and satisfies $G_{\ell-e+1}$, we have a complete characterization of the Cohen–Macaulayness of $\mathcal{R}(E)$ in terms of a presentation matrix of E (cf. Theorem 4.7) and, furthermore, if $\mathcal{R}(E)$ is Cohen–Macaulay we give the form of the canonical module of $\mathcal{R}(E)$ (cf. Proposition 4.10). One can say more if $R = k[X_1,\ldots,X_d]$ is a polynomial ring over a field, with $d \geq 2$, and E is a graded R-module of projective dimension one, having a linear syzygy matrix φ and satisfying G_d. In this case $\mathcal{R}(E)$ is Cohen–Macaulay and, moreover, its defining ideal in $R[T_1,\ldots,T_n]$ is generated by the quadratic forms $I_1((X_1,\ldots,X_d)\cdot B(\varphi))$ and by the $d \times d$ minors of $B(\varphi)$, where $B(\varphi)$ is the jacobian dual matrix of φ (see [43] for the latter notion).

For modules of projective dimension two a similar structure theorem is not available. Instead we obtain several theorems that may nevertheless indicate that in this context $\mathcal{R}(E)$ is yielding. For example, we generalize results of [49], [43] either by lifting the previously existing restriction on the local number of generators or on the second Betti number of E (cf. Proposition 4.12 and Theorem 4.17). These results typically address the question of the Cohen–Macaulayness of $\mathcal{R}(E)$ and the linear type property of E. Moreover, we add some material accessory to the "factorial conjecture" saying that if R is Gorenstein and if E is of linear type (always assumed to have projective dimension two) then $\mathcal{R}(E)$ is never quasi-Gorenstein (cf. Corollary 4.18). An amusing new result for ideals follows as a consequence: if I is an unmixed ideal of grade two, projective dimension at most two, normally torsionfree and of linear type then it is necessarily perfect.

In Section 5 we carry out a systematic investigation of Rees algebras of *ideal modules*, so called because of their similarity to ideals. These are torsionfree modules E having the property that the double dual E^{**} is free. Thus they afford a natural embedding into a free module of the same rank. They include ideals of grade at least two and direct sum of such ideals. Many vector bundles have their modules of global sections with this property.

Now for a nonfree ideal module E, we define its *deviation* (resp. *analytic deviation*) to be $d(E) := n-e+1-c$ (resp. $ad(E) := \ell-e+1-c$), where c stands for the codimension of the non free locus of E. Then $d(E) \geq ad(E) \geq 0$ holds, thus giving rise to a notion of complete intersection (resp. almost complete intersection, equimultiple) modules according to whether $d(E)=0$ (resp. $d(E)=1$, $ad(E)=0$). We show that a complete intersection module is of linear type and its Rees algebra is Cohen–Macaulay and that the same holds for an almost complete intersection module under a mild condition. Ideals of analytic deviation at most one have revealed ubiquitous in the recent literature (cf., e.g., [19]). Here we give, for ideal modules E satisfying $ad(E) \leq 1$, a version of one of the main theorems in the ideal case. Namely, it is shown that if, moreover, E has reduction number at most one and satisfies a condition on the local number of generators then $\mathcal{R}(E)$ is Cohen–Macaulay if and only if depth $E \geq \dim R - \ell + e$ (cf. Corollary 5.5).

The proofs of several of the results in this section are not only far from an obvious extension of the ideal case, they are not even won by an easy play of passing to the corresponding generic Bourbaki ideal as this may increase the deviation and the analytic deviation. One has rather to resort to an important device called the Artin–Nagata property AN_s ([46]). The ultimate role of this property is to control to some extent the Cohen–Macaulay property for Rees algebras of ideals. Now it turns out that if E is an ideal module locally free in codimension $s \geq 1$ then its generic Bourbaki ideal satisfies AN_s (cf. Proposition 5.3). This fact leads to another result of linkage nature as follows. Let U be a complete intersection module locally free on the punctured spectrum of R and let $E = U :_{U^{**}} \mathfrak{m}$ (the lift of the socle of U^{**}/U to the free module U^{**}). Then E is equimultiple with reduction number at most one and $\mathcal{R}(E)$ is Cohen–Macaulay (cf. Theorem 5.14). By using the Bourbaki ideal, we deduce this result from another theorem for ideals to the effect that socles modulo d-sequences give rise to ideals having reduction number one, but arbitrarily large analytic deviation (cf. Theorem 5.10).

The last part of the section deals with the Buchsbaum–Rim multiplicity as a tool for estimating the reduction number of ideal modules locally free in the punctured spectrum (cf. Corollary 5.18).

2 Basic Properties

In this section we review some definitions and general facts about Rees algebras of modules.

Let R be a Noetherian ring with total ring of quotients K and let E be a finitely generated R–module. One says that E has a *rank* if $K \otimes_R E \cong K^e$, in which case e is said to be the rank of E. By $\mathcal{S}(E)$ we denote the symmetric algebra of E. We will adopt the following definition of Rees algebras:

Definition 2.1 Let R be a Noetherian ring and E a finitely generated R–module having a rank. The *Rees algebra* $\mathcal{R}(E)$ of E is $\mathcal{S}(E)$ modulo its R-torsion submodule.

If E is a submodule of a free R-module G, some authors define the Rees algebra of E to be the image of the natural map $\mathcal{S}(E) \to \mathcal{S}(G)$ (see [40] for a similar approach). The two definitions coincide if the assumptions overlap, i.e., for a finitely generated torsionfree module having a rank (i.e., for a finitely generated module E such that $K \otimes_R E$ is K-free and the R-map $E \to K \otimes_R E$ is injective) as in this situation the kernel of $\mathcal{S}(E) \to \mathcal{S}(G)$ is the R-torsion submodule of $\mathcal{S}(E)$. The present definition depends solely on E, not on any particular embedding of E. Moreover, $\mathcal{R}(E)$ thus defined has the obvious universal property with respect to R-homomorphisms $E \to B$ into a torsionfree R-algebra B.

If $\mathcal{S}(E) = \mathcal{R}(E)$ then E is said to be of *of linear type*.

Proposition 2.2 *Let R be a Noetherian ring of dimension d and E a finitely generated R–module having a rank e. Then*

$$\dim \mathcal{R}(E) = d + e = d + \text{height } \ \mathcal{R}(E)_+.$$

Proof. We may assume that E is torsionfree, in which case E can be embedded into a free module $G = R^e$. Now $\mathcal{R}(E)$ is a subalgebra of the polynomial ring $S = \mathcal{R}(G) = R[t_1, \ldots, t_e]$. As in the case of ideals, the minimal primes of $\mathcal{R}(E)$ are exactly of the form $\mathfrak{P} = \mathfrak{p}S \cap \mathcal{R}(E)$, where $\mathfrak{p}$ ranges over all minimal primes of R. Write $\overline{R} = R/\mathfrak{p}$ and $\overline{E}$ for the image of E in $\overline{R} \otimes_R G$. Since $\mathcal{R}(E)/\mathfrak{P} \cong \mathcal{R}_{\overline{R}}(\overline{E})$, we may replace R and E by $\overline{R}$ and $\overline{E}$ to assume that R is a domain. But then the assertions follow from the dimension formula for graded domains ([51, 1.2.2]). ☐

In the setting of Definition 2.1 let $U \subset E$ be a submodule. One says that U is a *reduction* of E or, equivalently, E is *integral* over U if $\mathcal{R}(E)$ is integral over the R-subalgebra generated by U.

Alternatively, the integrality condition is expressed by the equations $\mathcal{R}(E)_{r+1} = U \cdot \mathcal{R}(E)_r$, $r \gg 0$. The least integer $r \geq 0$ for which this equality holds is called the *reduction number* of E *with respect to* U and denoted by $r_U(E)$. For any reduction U of E the module E/U is torsion, hence U has a rank and rank $U =$ rank E. This follows from the fact that a module of linear type such as a free module admits no proper reductions.

If R is moreover local with residue field k then the *special fiber* of $\mathcal{R}(E)$ is the ring $\mathcal{F}(E) = k \otimes_R \mathcal{R}(E)$; its Krull dimension is called the *analytic spread* of E and is denoted by $\ell(E)$.

Now assume in addition that k is infinite. A reduction of E is said to be *minimal* if it is minimal with respect to inclusion. For any reduction U of E one has $\nu(U) \geq \ell(E)$ ($\nu(\cdot)$ denotes the minimal number of generators function), and equality holds if and only if U is minimal. Minimal reductions arise from the following construction: The algebra $\mathcal{F}(E)$ is a standard graded algebra of dimension $\ell = \ell(E)$ over the infinite field k. Thus it admits a Noether normalization $k[y_1, \ldots, y_\ell]$ generated by linear forms; lift these linear forms to elements $x_1, \ldots, x_\ell$ in $\mathcal{R}(E)_1 = E$, and denote by U the submodule generated by $x_1, \ldots, x_\ell$. By Nakayama's Lemma, for all large r we have $\mathcal{R}(E)_{r+1} = U \cdot \mathcal{R}(E)_r$, making U a minimal reduction of E.

Having established the existence of minimal reductions, we can define the *reduction number* $r(E)$ of E to be the minimum of $r_U(E)$, where U ranges over all minimal reductions of E.

Proposition 2.3 *Let R be a Noetherian local ring of dimension $d \geq 1$ and let E be a finitely generated R–module having a rank e. Then*

$$e \leq \ell(E) \leq d + e - 1.$$

Proof. We may assume that the residue field of R is infinite. Let $\mathfrak{m}$ be the maximal ideal of R and U any minimal reduction of E. Now $e = \text{rank } E = \text{rank } U \leq \nu(U) = \ell(E)$. On the other hand, by the proof of Proposition 2.2, $\mathfrak{m}\mathcal{R}(E)$ is not contained in any minimal prime of $\mathcal{R}(E)$. Therefore $\ell(E) = \dim \mathcal{F}(E) \leq \dim \mathcal{R}(E) - 1 = d + e - 1$, where the last equality holds by Proposition 2.2. $\square$

3 Generic Bourbaki Ideals

By a *Bourbaki ideal* of a module E we mean an ideal I fitting into a *Bourbaki sequence*

$$0 \to F \longrightarrow E \longrightarrow I \to 0,$$

with F a free module (see [1, Chap. 7, §4, Théorème 6]). We wish to investigate how various properties relevant to the study of Rees algebras pass from E to I and vice versa. We may restrict ourselves to *generic* Bourbaki sequences, which, besides being easier to handle have the advantage of producing ideals which are more or less uniquely determined by the module E. For technical reasons it will be useful to consider generic Bourbaki ideals with respect to a submodule U, by which we mean that the free module F is a generic submodule of a fixed submodule U of E.

We first need several definitions and lemmas before establishing the existence of generic Bourbaki ideals in Proposition 3.2 and Definition 3.3. The main results of the section are summarized in Theorem 3.5.

Let R be a Noetherian ring and E a finitely generated R–module having a rank $e > 0$. We say that E satisfies G_s, s an integer, if $\nu(E_\mathfrak{p}) \leq \dim R_\mathfrak{p} + e - 1$ for every $\mathfrak{p} \in \text{Spec}(R)$ with $1 \leq \dim R_\mathfrak{p} \leq s - 1$. In terms of Fitting ideals this condition is equivalent to height $F_i(E) \geq i - e + 2$ for $e \leq i \leq e + s - 2$. We say that E satisfies $\widetilde{G}_s$ provided the conditions above hold with "depth" and "grade" in place of "dim" and "height", respectively. If G_s is satisfied for every s, E is said to satisfy G_∞ (or $\mathcal{F}_1$ in the terminology of [14]).

Lemma 3.1 *Let R be a Noetherian ring and $U = \sum_{i=1}^n Ra_i$ an R–module having a rank $e \geq 2$. Let $R' = R[z_1, \ldots, z_n]$ be a polynomial ring, $U' = R' \otimes_R U$, and $x = \sum_{i=1}^n z_i a_i \in U'$. Then $U'/(x)$ has rank $e - 1$. If U satisfies G_s or $\widetilde{G}_s$, then so does $U'/(x)$.*

Proof. Let φ be a matrix with n rows presenting U with respect to the generators $a_1, \ldots, a_n$, and consider the matrix

$$\Phi = \left[\begin{array}{c|c} z_1 & \\ \vdots & \quad\varphi\quad \\ z_n & \end{array} \right],$$

which presents $U'/(x)$ over R'.

We first show that for any integer t,

$$\text{height } I_{t+1}(\Phi) \geq \min\{\text{ height } I_t(\varphi), n-t\}. \tag{1}$$

$$\text{grade } I_{t+1}(\Phi) \geq \min\{\text{ grade } I_t(\varphi), n-t\}, \tag{2}$$

For this we may assume that $0 \leq t \leq n$. Let $\mathfrak{p}$ be any prime ideal of R' containing $I_{t+1}(\Phi)$. We have to prove that $\dim R'_{\mathfrak{p}} \geq \min\{\text{height } I_t(\varphi), n-t\}$ and depth $R'_{\mathfrak{p}} \geq \min\{\text{grade } I_t(\varphi), n-t\}$. This is clear if $I_t(\varphi) \subset \mathfrak{p}$. On the other hand, if $I_t(\varphi) \not\subset \mathfrak{p}$, then we may assume that the $t \times t$ minor Δ of φ defined by the first t rows and columns of φ is not contained in $\mathfrak{p}$, and it suffices to prove the assertion for R_Δ in place of R. Now, after elementary row operations and a linear change of variables in R', we may suppose that

$$\Phi = \left[\begin{array}{c|c|c} z_1 & 1_{t\times t} & \\ \vdots & & \star \\ z_n & 0 & \end{array}\right].$$

Hence $(z_{t+1}, \ldots, z_n) \subset I_{t+1}(\Phi) \subset \mathfrak{p}$, which yields $\dim R'_{\mathfrak{p}} \geq \text{depth } R'_{\mathfrak{p}} \geq n-t$, proving (1) and (2).

Notice that $I_{n-e+2}(\Phi) \subset R' I_{n-e+1}(\varphi) = 0$. Furthermore by (2),

$$\text{grade } I_{n-e+1}(\Phi) \geq \min\{\text{grade } I_{n-e}(\varphi), e\} \geq 1.$$

Thus $U'/(x)$ has rank $e-1$. If U satisfies G_s then (1) yields for every i, $e-1 \leq i \leq e-1+s-3$,

$$\begin{aligned} \text{height } F_i(U'/(x)) &\geq \min\{\text{height } F_{i+1}(U), i+1\} \\ &\geq \min\{i+1-e+2, i+1\} \geq i-(e-1)+2, \end{aligned}$$

because $e \geq 2$. Thus G_s holds for $U'/(x)$ as well. The proof of the corresponding claim for $\widetilde{G}_s$ is likewise. □

Proposition 3.2 *Let R be a Noetherian ring and E a finitely generated torsionfree R–module having a rank $e > 0$. Let $U = \sum_{i=1}^n Ra_i$ be a submodule of E. Assume that E satisfies $\widetilde{G}_2$ and that* grade $E/U \geq 2$. *Further let $R' = R[\{z_{ij}, 1 \leq i \leq n,\ 1 \leq j \leq e-1\}]$ be a polynomial ring. Set*

$$U' = R' \otimes_R U, \quad E' = R' \otimes_R E, \quad x_j = \sum_{i=1}^n z_{ij} a_i \in U', \quad F = \sum_{j=1}^{e-1} R' x_j.$$

(a) *F is a free R'–module of rank $e-1$ and $E'/F \cong I$ for some R'–ideal I. Let J denote the image of U'/F in I. Then* grade $J > 0$ *and $J_{\mathfrak{p}} = I_{\mathfrak{p}} \cong R'_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ with* depth $R'_{\mathfrak{p}} \leq 1$.

(b) *Assume that* height $\operatorname{ann}(E/U) \geq s$. *Then I satisfies G_s if and only if J does if and only if E does. The same holds for $\widetilde{G}_s$ assuming that grade $E/U \geq s$.*

(c) *I can be chosen to have grade* ≥ 2 *if and only if* E *is orientable. In this case,* grade $J \geq 2$.

Proof. (a): First notice that rank $U = e$ and that U satisfies $\widetilde{G}_2$. If $e \geq 2$ we apply Lemma 3.1 repeatedly to conclude that U'/F has rank 1. Thus F has rank $e-1$ and hence is free. Furthermore, again by Lemma 3.1, U'/F satisfies $\widetilde{G}_2$. Now as grade $E'/U' \geq 2$, it follows that E'/F has rank 1 and satisfies $\widetilde{G}_2$. On the other hand, F is free and E' is torsionfree over R'. Therefore E'/F is torsionfree, and hence can be identified with an R'–ideal I that satisfies $\widetilde{G}_2$ and has grade > 0. The assertions about J follow immediately since $I/J \cong E'/U'$ is an R'–module of grade ≥ 2.

(b): If $I \cong E'/F$ satisfies G_s then E obviously has the same property. Conversely, assume that E satisfies G_s. In this case U satisfies G_s since height $\operatorname{ann}(E/U) \geq s$, and therefore J has the same property by Lemma 3.1. Thus I is G_s, again because $\operatorname{ann}(I/J) = \operatorname{ann}(E'/U')$ has height $\geq s$. The proof of the corresponding statement for $\widetilde{G}_s$ is similar.

(c): Since I satisfies $\widetilde{G}_2$, I is isomorphic to an R'–ideal of grade ≥ 2 if and only if $I^{**} \cong R'$, which means that I is orientable. Computing determinants along the exact sequence

$$0 \to F \longrightarrow E' \longrightarrow I \to 0,$$

where F is free and I satisfies $\widetilde{G}_2$, one sees that I is orientable if and only if E' or, equivalently, E is orientable. Again the assertion about J is clear since grade $I/J \geq 2$. $\square$

Let $(R, \mathfrak{m})$ be a Noetherian local ring. If $\mathbf{Z}$ is a set of indeterminates over R, we denote by $R(\mathbf{Z})$ the localization $R[\mathbf{Z}]_{\mathfrak{m}R[\mathbf{Z}]}$. Let E a finitely generated R–module having a rank $e > 0$ and $U = \sum_{i=1}^{n} Ra_i$ a submodule of E such that E/U is a torsion module. Further let z_{ij}, $1 \leq i \leq n$, $1 \leq j \leq e-1$, be indeterminates

$$R'' = R(\{z_{ij}\}), \quad U'' = R'' \otimes_R U, \quad E'' = R'' \otimes_R E, \quad x_j = \sum_{i=1}^{n} z_{ij} a_i \in E'',$$

and $F = \sum_{j=1}^{e-1} R'' x_j$ (which, by Lemma 3.1, is a free R''–module of rank $e-1$). Assume that E''/F is torsionfree over R'' (which, by Proposition 3.2(a), holds if E is a torsionfree module satisfying $\widetilde{G}_2$ and grade $E/U \geq 2$). In this case $E''/F \cong I$ for some R''–ideal I with grade $I > 0$. We will denote the image of U''/F in I by J.

Definition 3.3 We call an R''–ideal I with $I \cong E''/F$ a *generic Bourbaki ideal of* E *with respect to* U. If $U = E$, we simply talk about a *generic Bourbaki ideal of* E and write $I = I(E)$.

Remark 3.4 (a) A generic Bourbaki ideal of E with respect to U is essentially unique. Indeed, suppose $I \subset R(Z)$ and $K \subset R(Y)$ are two such ideals defined using generating sequences $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ of U, and sets of variables $Z = \{z_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq e-1\}$ and $Y = \{y_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq e-1\}$, respectively; then there exists an automorphism φ of the R–algebra $S = R(Y, Z)$ and a unit u of $\operatorname{Quot}(S)$ so that

$$\varphi(IS) = uKS.$$

Furthermore, $u=1$ if I and K have grade ≥ 2.

(b) With the notation preceding Definition 3.3, $J = I(U)$.

(c) If $E \cong R^{e-1} \oplus L$, for some R–ideal L, then $LR'' = I(E)$.

(d) (See also Proposition 3.2(c)) Conversely, if $I = I(E)$ has grade ≥ 3, then $E \cong R^{e-1} \oplus L$ for some R–ideal L.

Proof. The proof of (a) is straightforward (see also [18, the proof of Proposition 1] or [24, the proof of 2.11.b]). Parts (b) and (c) follow immediately from the definition of generic Bourbaki ideals. As for (d), notice that if grade $I \geq 3$ then $\mathrm{Ext}^1_{R''}(I,F) = 0$, thus $E'' \cong F \oplus I$, which induces an R''–epimorphism $\wedge^{e-1}E'' \to R''$. Since R is local, there exists an R–epimorphism $\wedge^{e-1}E \to R$ and then E has an R–free summand of rank $e-1$. ☐

We mention in passing how (generic) Bourbaki ideals can be obtained explicitly from the presentation matrix of a (not necessarily torsionfree) module. Let R be a Noetherian local ring, let E a finitely generated R-module having a rank $e>0$, and let $F = \sum_{j=1}^{e-1} R''x_j$ be defined as above for $U = E$. Extend the basis of F to a generating sequence $x_1, \ldots, x_n$ of E'' and let φ be a matrix presenting E'' with respect to these generators. Finally let ψ be the $n-e+1$ by $n-e$ submatrix of φ consisting of the last rows and columns of φ, and write $I = I_{n-e}(\psi)$. After elementary column operations on φ one may assume that grade $I>0$. Now I is indeed a generic Bourbaki ideal of $E/\tau_R(E)$, provided this module satisfies $\widetilde{G}_2$. Here $\tau_R(\cdot)$ denotes the torsion functor.

Having defined the Bourbaki ideal I of a module E we now embark on a comparison of their Rees algebras.

Theorem 3.5 *Let R be a Noetherian local ring and E a finitely generated R–module having a rank $e>0$. Let U be a reduction of E. Further let $I \cong E''/F$ be a generic Bourbaki ideal of E with respect to U and write J for the image of U''/F in I.*

(a) (i) *$\mathcal{R}(E)$ is Cohen–Macaulay if and only if $\mathcal{R}(I)$ is Cohen–Macaulay.*

(ii) (*In case R is universally catenary*) *$\mathcal{R}(E)$ is normal with* depth $\mathcal{R}(E) \otimes_R R_{\mathfrak{p}} \geq e+1$ *for every nonzero prime $\mathfrak{p}$ of R if and only if $\mathcal{R}(I)$ is normal.*

(iii) *E is of linear type with* grade $\mathcal{R}(E)_+ \geq e$ *if and only if I is of linear type.*

(b) *If any of the conditions* (i), (ii), (iii) *of* (a) *hold, then $\mathcal{R}(E'')/(F) \cong \mathcal{R}(I)$ and the generators $x_1, \ldots, x_{e-1}$ of F form a regular sequence on $\mathcal{R}(E'')$.*

(c) *If $\mathcal{R}(E'')/(F) \cong \mathcal{R}(I)$, then*

(i) *J is a reduction of I and* $\mathrm{r}_U(E) = \mathrm{r}_J(I)$*;*

(ii) (*In case the residue field of R is infinite and $U = E$*) $\mathrm{r}(E) = \mathrm{r}(I)$.

We are going to prove this theorem through a series of separate results that are stronger than Theorem 3.5

By Lemma 3.6, x is regular on $\mathcal{R}''$. Thus it suffices to prove that $K = 0$. As K is an R''–torsion module by Lemma 3.6, we may suppose that $\dim R'' \geq 1$. Furthermore, after localizing R if needed we may assume that K vanishes locally on the punctured spectrum of R and hence of R''. On the other hand by Lemma 3.6, K is annihilated by some power of $U\overline{\mathcal{R}}$, and hence by some power of $E\overline{\mathcal{R}} = \overline{\mathcal{R}}_+$ because E is integral over U. Thus $K \subset H^0_M(\overline{\mathcal{R}})$, and it remains to show that $H^0_M(\overline{\mathcal{R}}) = 0$.

As $\dim R'' \geq 1$ and $\operatorname{rank} \overline{E} = e-1 \geq 1$, Proposition 2.1 implies $\dim \mathcal{R}(\overline{E}) = \dim R'' + \operatorname{rank} \overline{E} \geq 2$, and hence by the S_2 assumption, depth $\mathcal{R}(\overline{E}) \geq 2$. Now the exact sequence

$$0 \to K \longrightarrow \overline{\mathcal{R}} \longrightarrow \mathcal{R}(\overline{E}) \to 0$$

with $K \subset H^0_M(\overline{\mathcal{R}})$, shows that $H^1_M(\overline{\mathcal{R}}) = H^1_M(\mathcal{R}(\overline{E})) = 0$. Thus the sequence

$$0 \to \mathcal{R}''(-1) \xrightarrow{\ x\ } \mathcal{R}'' \longrightarrow \overline{\mathcal{R}} \to 0$$

induces an exact sequence

$$0 \to H^0_M(\overline{\mathcal{R}}) \longrightarrow H^1_M(\mathcal{R}'')(-1) \xrightarrow{\ x\ } H^1_M(\mathcal{R}'') \longrightarrow H^1_M(\overline{\mathcal{R}}) = 0.$$

If we knew that $H^1_M(\mathcal{R}'')$ is finitely generated, then by the graded version of Nakayama's Lemma, $H^1_M(\mathcal{R}'') = 0$ and therefore $H^0_M(\overline{\mathcal{R}}) = 0$.

So it remains to verify that $H^1_M(\mathcal{R}'')$ is finitely generated. As is well known, by the graded version of local duality, $H^1_M(\mathcal{R}'')$ is finitely generated if and only if $\widehat{R''} \otimes_{R''} \mathcal{R}''$ has no one–dimensional associated primes. To verify the latter write R, E, $\mathcal{R}$ instead of $\widehat{R''}$, $\widehat{R''} \otimes_{R''} E''$, $\widehat{R''} \otimes_{R''} \mathcal{R}'' = \mathcal{R}(\widehat{R''} \otimes_{R''} E'')$, and notice that by flatness E still has a rank $e \geq 2$. With this assumption we show that for every associated prime P of the ring $\mathcal{R}$, $\dim P \geq 2$. However, $\mathcal{R}$ being torsionfree over R, the contraction $\mathfrak{p} = P \cap R$ is an associated prime of R. Hence $E_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$–module of rank e, and so $\mathcal{R}_{\mathfrak{p}} \cong R_{\mathfrak{p}}[t_1, \ldots, t_e]$ is a polynomial ring in e variables. Therefore $P_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}[t_1, \ldots, t_e]$, which implies $\dim P \geq \dim P_{\mathfrak{p}} = e \geq 2$. □

Proposition 3.8 *Let R be a Noetherian ring and E a finitely generated R–module having a rank ≥ 2. Let $U = \sum_{i=1}^n Ra_i$ be a reduction of E. Further let R', E', x, and $\overline{E}$ be defined as in* Lemma 3.6.

(a) *If* grade $\mathcal{R}(E)_+ \geq 2$ *then $\mathcal{R}(E')/(x) \cong \mathcal{R}(\overline{E})$ and x is regular on $\mathcal{R}(E')$.*

(b) *If $\mathcal{R}(E)$ is normal and* depth $\mathcal{R}(E) \otimes_R R_{\mathfrak{p}} \geq 3$ *for every prime $\mathfrak{p} \neq 0$, then $\mathcal{R}(\overline{E})$ is normal.*

Proof. Write $\mathcal{R} = \mathcal{R}(E)$, $\mathcal{R}' = \mathcal{R}(E')$, $\overline{\mathcal{R}} = \mathcal{R}'/(x)$, and notice that $\sqrt{\overline{\mathcal{R}}_+} = \sqrt{U\overline{\mathcal{R}}}$.

(a): Let K be the kernel of the natural epimorphism $\overline{\mathcal{R}} \to \mathcal{R}(\overline{E})$. By Lemma 3.6, x is regular on $\mathcal{R}'$. Thus grade $U\overline{\mathcal{R}} =$ grade $\overline{\mathcal{R}}_+ > 0$. On the other hand, by the same lemma, $K = H^0_{U\overline{\mathcal{R}}}(\overline{\mathcal{R}})$. Hence $K = 0$.

(b): Notice that height $\mathcal{R}_+ = e \geq 2$ by Proposition 2.2, hence grade $\mathcal{R}_+ \geq 2$. Thus by part (a), $\overline{\mathcal{R}} = \mathcal{R}(E')/(x)$ and x is regular on $\mathcal{R}'$. In the proof of Lemma 3.6 we have already seen that for every

i, $1 \leq i \leq n$, $\overline{\mathcal{R}}_{a_i}$ is a ring of fractions of a polynomial ring over $\mathcal{R}$, hence $\overline{\mathcal{R}}_{a_i}$ is normal. Therefore it suffices to examine the localizations $\overline{\mathcal{R}}_{\mathfrak{q}}$ with $\mathfrak{q} \in V(U\mathcal{R}') = V(\mathcal{R}'_+)$. Now $\mathfrak{q} = (\mathfrak{p}', \mathcal{R}'_+)$, where $\mathfrak{p}' = \mathfrak{q} \cap R'$. First assume that $\dim \overline{\mathcal{R}}_{\mathfrak{q}} \leq 1$. In this case $\mathfrak{p}' = 0$ and then $\overline{\mathcal{R}}_{\mathfrak{q}}$ is regular. Next suppose that $\dim \overline{\mathcal{R}}_{\mathfrak{q}} \geq 2$. We need to show that depth $\overline{\mathcal{R}}_{\mathfrak{q}} \geq 2$. So write $\mathfrak{p} = \mathfrak{q} \cap R$ and notice that $\mathcal{R}_{(\mathfrak{p}, \mathcal{R}_+)} \hookrightarrow \mathcal{R}'_{\mathfrak{q}}$ is a flat local homomorphism whose closed fiber is regular. If $\mathfrak{p} = 0$, then $\mathcal{R}_{(\mathfrak{p}, \mathcal{R}_+)}$ is regular, thus $\mathcal{R}'_{\mathfrak{q}}$ is regular, and therefore $\overline{\mathcal{R}}_{\mathfrak{q}}$ is Cohen–Macaulay. If on the other hand $\mathfrak{p} \neq 0$, then depth $\mathcal{R} \otimes_R R_{\mathfrak{p}} \geq 3$ by our assumption, hence

$$\text{depth } \mathcal{R}'_{\mathfrak{q}} \geq \text{depth } \mathcal{R}_{(\mathfrak{p}, \mathcal{R}_+)} = \text{depth } \mathcal{R} \otimes_R R_{\mathfrak{p}} \geq 3,$$

which again implies that depth $\overline{\mathcal{R}}_{\mathfrak{q}} \geq 2$. □

Lemma 3.9 *Let R be a Noetherian local ring with infinite residue field and $E = \sum_{i=1}^{n} Ra_i$ an R–module having a rank $e > 0$. Let R'', E'', and $x_1, \ldots, x_{e-1}$ be defined as in* Definition 3.3 *with $U = E$. Then $x_1, \ldots, x_{e-1}$ form part of a minimal generating set of a minimal reduction V of E'' with* $\mathrm{r}_V(E'') = \mathrm{r}(E)$.

Proof. The assertion follows as in the case of ideals, see [45, the proof of 3.4]. □

Proposition 3.10 *Let R be a Noetherian local ring and E a finitely generated R–module having a rank ≥ 2. Let $U = \sum_{i=1}^{n} Ra_i$ be a reduction of E. Further let $\overline{E}$ be defined as in* Theorem 3.7.

(a) $\ell(\overline{E}) = \ell(E) - 1$.

(b) (*In case the residue field of R is infinite and $U = E$*) $r(\overline{E}) \leq r(E)$.

Proof. (a): With the notation of Theorem 3.7 let k be the residue field of R'' and write $\mathcal{F}(\cdot) = k \otimes_{R''} \mathcal{R}(\cdot)$. Now $U\mathcal{F}(E'')$ is primary to the maximal ideal $\mathcal{F}(E'')_+$, which has height $\geq$ rank $E \geq 2$ by Proposition 2.3. Thus the image of x in $\mathcal{F}(E'')$ forms part of a homogeneous system of parameters. Furthermore by Lemma 3.6, the kernel of the natural epimorphism from $\overline{\mathcal{F}} = \mathcal{F}(E'')/(x)$ onto $\mathcal{F}(\overline{E})$ is contained in $H^0_{U\overline{\mathcal{F}}}(\overline{\mathcal{F}}) = H^0_{\overline{\mathcal{F}}_+}(\overline{\mathcal{F}})$. Therefore

$$\ell(\overline{E}) = \dim \mathcal{F}(\overline{E}) = \dim \overline{\mathcal{F}} = \dim \mathcal{F}(E'') - 1 = \ell(E) - 1.$$

(b): Let V be a minimal reduction of E'' as in Lemma 3.9, and write $\overline{V}$ for the image of V in $\overline{E}$. Now by (a) and Lemma 3.9, $\overline{V}$ is a minimal reduction of $\overline{E}$. Therefore $r(\overline{E}) \leq r_{\overline{V}}(\overline{E})$. But $r_{\overline{V}}(\overline{E}) \leq r_V(E) = r(E)$, where the last equality follows again from Lemma 3.9. □

The next proposition deals with not necessarily generic Bourbaki ideals I of a module E. It clarifies as to when $\mathcal{R}(E)$ yields a *deformation* of $\mathcal{R}(I)$, by which we mean that the kernel of the natural epimorphism $\mathcal{R}(E) \twoheadrightarrow \mathcal{R}(I)$ can be generated by a regular sequence of $\mathcal{R}(E)$. Part (b) of the proposition has been known if R is a domain ([15], [21]).

Proposition 3.11 *Let R be a Noetherian ring and E a finitely generated R–module having a rank $e > 0$. Further let $I \cong E/F$ be a Bourbaki ideal of E, where F is a free module with basis $x_1, \ldots x_{e-1}$.*

(a) *The following are equivalent:*

(i) *$\mathcal{R}(E)/(F)$ is R–torsionfree;*

(ii) *$\mathcal{R}(E)/(F) \cong \mathcal{R}(I)$;*

(iii) *$\mathcal{R}(E)/(F) \cong \mathcal{R}(I)$ and $x_1, \ldots, x_{e-1}$ form a regular sequence on $\mathcal{R}(E)$;*

(iv) *$\mathcal{R}(E)$ is a deformation of $\mathcal{R}(I)$.*

(b) *If I is of linear type, then so is E and the equivalent conditions of* (a) *hold.*

Proof. (a): (i) $\Rightarrow$ (ii): This holds since there are canonical epimorphisms $\mathcal{S}(I) \cong \mathcal{S}(E)/(F) \twoheadrightarrow \mathcal{R}(E)/(F) \twoheadrightarrow \mathcal{R}(I)$.

(ii) $\Rightarrow$ (iii): Write $\mathfrak{a} = (F) \subset S = \mathcal{R}(E)$. Proving that $x_1, \ldots, x_{e-1}$ form an S–regular sequence is equivalent to showing that the natural epimorphism

$$\varphi : S/\mathfrak{a}[Y_1, \ldots, Y_{e-1}] \twoheadrightarrow \mathrm{gr}_{\mathfrak{a}}(S)$$

is an isomorphism ([36], Theorem 27). Now locally at every associated prime $\mathfrak{p}$ of R, the embedding $F \hookrightarrow E$ splits, hence $x_1, \ldots, x_{e-1}$ form a regular sequence on $R_{\mathfrak{p}} \otimes_R S$, and therefore $\mathrm{id}_{R_{\mathfrak{p}}} \otimes \varphi$ is an isomorphism. This shows that $\ker \varphi$ is R–torsion. But then $\ker \varphi$ vanishes since $S/\mathfrak{a} \cong \mathcal{R}(I)$ is R–torsionfree.

(iii) $\Rightarrow$ (iv): This is clear.

(iv) $\Rightarrow$ (i): Let $\mathfrak{b}$ denote the kernel of the natural epimorphism $\mathcal{R}(E) \twoheadrightarrow \mathcal{R}(I)$. Obviously $[\mathfrak{b}]_0 = 0$ and $[\mathfrak{b}]_1 = F$. On the other hand by assumption (iv) and Proposition 2.2, $\mathfrak{b}$ is generated by $e - 1 = \mathrm{rank}\, F$ elements. Therefore $\mathfrak{b} = (F)$, and $\mathcal{R}(E)/(F) \cong \mathcal{R}(I)$ is R–torsionfree.

(b): Since $\mathcal{S}(E)/(F) \cong \mathcal{S}(I)$ is R–torsionfree it follows that $\mathcal{R}(E)/(F)$ is R–torsionfree as well. Thus the conditions of (a) hold. Now write $\mathfrak{a} = (F) \subset S = \mathcal{S}(E)$. Again by the R–torsionfreeness of $S/\mathfrak{a}$, an argument as in the proof of (a) shows that $\mathrm{gr}_{\mathfrak{a}}(S) \cong S/\mathfrak{a}[Y_1, \ldots, Y_{e-1}]$. Thus $\mathrm{gr}_{\mathfrak{a}}(S)$ is R–torsionfree, and then S has the same property since $\bigcap_{i \geq 0} \mathfrak{a}^i = 0$. □

We are now ready to assemble the proof of Theorem 3.5.

Proof of Theorem 3.5: For parts (a) and (b) we use induction on e, successively factoring out the elements $x_1, \ldots, x_{e-1}$. By Lemma 3.1, this does not change our assumption on E. We may assume that $e \geq 2$, in which case height $\mathcal{R}(E)_+ \geq 2$ (see the proof of Proposition 2.1).

(a): Assertion (i) is a consequence of Proposition 3.8(a) and Theorem 3.7. Part (ii) follows from Proposition 3.8, Theorem 3.7, and the fact that normality deforms to a catenary ring. Finally for (iii) one uses Propositions 3.8(a) and 3.11.

(b): This follows from Proposition 3.8(a).

(c): To see (i), notice that J is a reduction of I. Moreover, $r_U(E) = r_J(I)$ because $\mathcal{R}(I)/J\mathcal{R}(I) \cong \mathcal{R}(E'')/U\mathcal{R}(E'')$. As to (ii), let L be a minimal reduction of I with $\mathrm{r}(I) = \mathrm{r}_L(I)$, and let W be the preimage of L in E''. As above one sees that $\mathrm{r}_L(I) = \mathrm{r}_W(E'')$, and since W is a minimal reduction of E'' by (i), one has $\mathrm{r}_W(E'') \geq \mathrm{r}(E)$. Thus $\mathrm{r}(I) \geq \mathrm{r}(E)$. The opposite inequality, follows from Proposition 3.10(b). □

4 Applications

In this section we use our technique of Bourbaki ideals to derive numerical measures on the Rees algebras of several classes of modules.

Bounds on the analytic spread and the reduction number

For our first application, the use of Bourbaki ideals is not strictly necessary, but it simplifies matters.

Proposition 4.1 *Let R be a Noetherian local ring and let E be a finitely generated torsionfree R–module having a rank $e > 0$.*

(a) *If E satisfies $\widetilde{G}_2$ and is not free, then $\ell(E) \geq e+1$.*

(b) *If E satisfies $\widetilde{G}_3$ and is orientable, and $\nu(E) \geq e+2$, then $\ell(E) \geq e+2$.*

Proof. By Proposition 3.2(a), E has a generic Bourbaki ideal $I = I(E)$ as in Definition 3.3. According to Proposition 3.2, we may assume that I satisfies $\widetilde{G}_2$, grade $I \geq 1$, $\nu(I) > 1$ for (a), and that I satisfies $\widetilde{G}_3$, grade $I \geq 2$, $\nu(I) > 2$ for (b). On the other hand by Proposition 3.10(a), $\ell(I) = \ell(E) - e + 1$. Hence it suffices to verify that $\ell(I) > 1$ or $\ell(I) > 2$, respectively. But this is a consequence of the known fact that if I is any R–ideal of grade > 0 satisfying $\widetilde{G}_{s+1}$ and $\ell(I) \leq s \leq$ grade I, then necessarily $\nu(I) \leq s$ (see [9]). □

Theorem 4.2 *Let R be a Cohen–Macaulay local ring of dimension $d > 0$ with infinite residue field and E a finitely generated R–module having a rank e. If $\mathcal{R}(E)$ is Cohen–Macaulay then*

$$r(E) \leq \ell(E) - e \leq d-1.$$

Proof. The rightmost inequality has been proved in Proposition 2.3. For the other inequality, we may replace E by $E/\tau(E) = [\mathcal{R}(E)]_1$ to assume that E is torsionfree. We may further assume that $e > 0$. Now let $F \subset E''$ be the R''–modules defined prior to Definition 3.3 with $U = E$, and write $\overline{E} = E''/F$. If $F \neq 0$, then $e \geq 2$ and hence grade $\mathcal{R}(E)_+ \geq 2$ by Proposition 2.2. Therefore Proposition 3.8(a) implies that $\mathcal{R}(\overline{E}) \cong \mathcal{R}(E'')/(F)$, which yields

$$\overline{E} \cong E''/F \cong [\mathcal{R}(E'')/(F)]_1 \cong [\mathcal{R}(\overline{E})]_1 \cong \overline{E}/\tau(\overline{E}).$$

Thus $\overline{E}$ is torsionfree over R''. Hence $\overline{E}$ is isomorphic to a generic Bourbaki ideal $I = I(E)$ as in Definition 3.3, where grade $I > 0$. By Theorem 3.5(a.i),(c.ii), $\mathcal{R}(I)$ is Cohen–Macaulay and $r(I) = r(E)$, whereas by Proposition 3.10(a), $\ell(I) = \ell(E) - e + 1$. In the case of a (not necessarily proper) ideal however, [27] shows that the Cohen–Macaulayness of the Rees algebra yields $r(I) \leq \ell(I) - 1$. Therefore $r(E) \leq \ell(E) - e$. $\square$

Corollary 4.3 *Let R be a Cohen–Macaulay ring and let E be a finitely generated torsionfree R–module having a rank. If $\mathcal{R}(E)$ is Cohen–Macaulay then E is free locally in codimension one.*

Proof. We may assume that R is a Cohen–Macaulay local ring of dimension $d = 1$ with infinite residue field. Now by Theorem 4.2, $r(E) = \ell(E) - \text{rank } E = 0$. Thus $\nu(E) = \ell(E) = \text{rank } E$, showing that E is free. $\square$

Small dimensions

In this section we apply the results about Bourbaki ideals provided in Theorem 3.5 to study the Rees algebras of modules of low projective dimensions (≤ 2) or low Krull dimensions (≤ 5).

Proposition 4.4 *Let R be a Cohen–Macaulay local ring of dimension* 2 *with infinite residue field and let E be a finitely generated torsionfree R–module having a rank.*

(a) (see also [31, 3.2]) $\mathcal{R}(E)$ *is Cohen–Macaulay if and only if E is free locally in codimension* 1 *and* $r(E) \leq 1$.

(b) *E is of linear type if and only if E satisfies* G_∞.

Proof. We may assume that $e = \text{rank } E > 0$.

(a): In the light of Theorem 4.2 and Corollary 4.3 we only need to show that $\mathcal{R}(E)$ is Cohen–Macaulay if E is locally free in codimension one and $r(E) \leq 1$. By Proposition 3.2(a), there exists a generic Bourbaki ideal $I = I(E)$ as in Definition 3.3 that has grade > 0 and is principal locally in codimension one. According to Theorem 3.5(a.i) it suffices to prove that $\mathcal{R}(I)$ is Cohen–Macaulay. Now if $\ell(I) = 1$, then I is principal ([9]), and the assertion is obvious. Otherwise $\ell(I) = 2$, and $r(I) \leq r(E) \leq 1$ by Proposition 3.10(b). But then the Cohen–Macaulayness of $\mathcal{R}(I)$ follows from [19, 2.1] (or [28, 3.4]).

(b): By [14] it suffices to show that E is of linear type in case G_∞ holds. According to Proposition 3.2(a),(b), E has a generic Bourbaki ideal $I \subset R''$ with grade $I > 0$ that satisfies G_∞. As $\dim R'' = 2$, I is an almost complete intersection and hence it is of linear type ([14]). Thus E is of linear type by Theorem 3.5(a.iii). $\square$

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension 2 and let $\mathfrak{a}_1, \mathfrak{a}_2$ be $\mathfrak{m}$–primary ideals. One says that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ have *joint reduction number zero* and writes $\mathrm{r}(\mathfrak{a}_1|\mathfrak{a}_2) = 0$, if there exist elements $a_i \in \mathfrak{a}_i$ so that $\mathfrak{a}_1\mathfrak{a}_2 = a_2\mathfrak{a}_1 + a_1\mathfrak{a}_2$.

Corollary 4.5 ([12, 3.4], see also [53, 3.1 and 3.6]) *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension 2 with infinite residue field and let $\mathfrak{a}_i$, $1 \le i \le e$, be $\mathfrak{m}$–primary ideals so that $\mathrm{r}(\mathfrak{a}_i|\mathfrak{a}_j) = 0$ for every i, j. Write $E = \oplus_{i=1}^{e}\mathfrak{a}_i$. Then $\mathcal{R}(E)$ is Cohen–Macaulay.*

Proof. Letting a_i, b_i be sufficiently general elements in $\mathfrak{a}_i$, one has $\mathfrak{a}_i^2 = (a_i, b_i)\mathfrak{a}_i$ for every i and $\mathfrak{a}_i\mathfrak{a}_j = a_j\mathfrak{a}_i + a_i\mathfrak{a}_j$ for every $i \neq j$. In $E = \oplus_{i=1}^{e}\mathfrak{a}_i\varepsilon_i$ consider the elements $x_i = a_i\varepsilon_i$ and $y = \sum_{i=1}^{e} b_i\varepsilon_i$, and let U be the submodule generated by $x_1,\ldots,x_e,y$. One immediately sees that U is a reduction of E with $\mathrm{r}_U(E) \le 1$. Hence $\mathrm{r}(E) \le 1$ because $\ell(E) \ge e+1$ by Proposition 4.1(a). Now the assertion follows from Proposition 4.4(a). ☐

The assumptions of the above corollary are satisfied if R is pseudo–rational and the ideals $\mathfrak{a}_i$ are integrally closed ([53, 3.5]).

For the proof of the next proposition, we need to recall the Artin–Nagata property AN_s. Let R be a Cohen–Macaulay local ring, I an R–ideal, and s an integer. We say that I satisfies AN_s if $R/J : I$ is Cohen–Macaulay for every ideal $J \subsetneq I$ with $\nu(J) \le i \le \mathrm{height}\, J : I$ and every $i \le s$.

Proposition 4.6 *Let R be a Cohen–Macaulay local ring of dimension d with infinite residue field and let E be a finitely generated torsionfree orientable R–module that satisfies G_d.*

(a) *Assume $d = 3$. If $r(E) \le 1$ then $\mathcal{R}(E)$ is Cohen–Macaulay. If E satisfies G_∞ then E is of linear type.*

(b) *Assume $d \le 4$. If $r(E) \le 2$ and* depth $E \ge 2$*, then $\mathcal{R}(E)$ is Cohen–Macaulay.*

(c) *Assume $d = 5$. If R is Gorenstein, $r(E) \le 2$, and* depth $E \ge 4$*, then $\mathcal{R}(E)$ is Cohen–Macaulay.*

Proof. By Proposition 3.2, E has a generic Bourbaki ideal $I \subset R''$ with grade $I \ge 2$ that satisfies G_d, and satisfies G_∞ if E does.

Now part (a) follows by the same argument as in the proof of Proposition 4.4. As for (b) and (c) we will show that $\mathcal{R}(I)$ is Cohen-Macaulay, which by Theorem 3.5(a.i) implies the Cohen-Macaulayness of $\mathcal{R}(E)$. First notice that depth $R/I \ge 1$ because depth $E \ge 2$. Furthermore, I satisfies AN_{d-3}. In the case of (b) this is obvious since $d - 3 <$ grade I, and in (c) the condition holds since depth $E \ge 4$ and thus R/I is Cohen-Macaulay. Now if $\ell(I) \le d-1$, then applying [46, 4.1.b] (with $s = d-1$) yields the Cohen-Macaulayness of $\mathcal{R}(I)$. If on the other hand $\ell(I) = d$, then the Cohen-Macaulayness of $\mathcal{R}(I)$ follows from [11, 1.1], since $r(I) \le r(E) \le 2$ by Proposition 3.10(b). ☐

Projective dimension one

Theorem 4.7 *Let R be a Gorenstein local ring with infinite residue field and let E be a finitely generated R–module with* proj dim $E = 1$*. Write $e =$* rank E*, let $s \ge e$ be an integer, and assume that E satisfies G_{s-e+1} and is torsionfree locally in codimension* 1*. Further let φ be a matrix presenting E with respect to a generating sequence $a_1,\ldots,a_n$ where $n \ge s$.*

(a) *The following are equivalent:*

(i) $\mathcal{R}(E)$ *is Cohen–Macaulay and* $\ell(E) \leq s$*;*

(ii) $r(E) \leq \ell(E) - e \leq s - e$*;*

(iii) $r(E_{\mathfrak{p}}) \leq s - e$ *for every prime* $\mathfrak{p}$ *with* $\dim R_{\mathfrak{p}} = \ell(E_{\mathfrak{p}}) - e + 1 = s - e + 1$*, and* $\ell(E) \leq s$*;*

(iv) *after elementary row operations,* $I_{n-s}(\varphi)$ *is generated by the maximal minors of the last* $n-s$ *rows of* φ*;*

(v) *after changing the generating sequence,* $F_s(E) = F_0(E/U)$ *for* $U = \sum_{i=1}^{s} Ra_i$*.*

(b) *If the equivalent conditions of* (a) *hold then U is a reduction of E with* $\mathrm{r}_U(E) = \mathrm{r}(E)$*. Furthermore, E is of linear type, or else,* $\ell(E) = s$ *and* $\mathrm{r}(E) = s - e$*.*

Proof. Notice that E is free locally in codimension one and torsionfree. We may suppose that E is not free. Let $I = I(E)$ be a generic Bourbaki ideal of E. By Proposition 3.2, we may assume I to be a perfect ideal of grade 2 satisfying G_{s-e+1}, and by Proposition 3.10(a), $\ell(I) = \ell(E) - e + 1$.

(a): The equivalence of (iv) and (v) is clear, (i) implies (ii) by Theorem 4.2, and (ii) obviously gives (iii). If E and s are replaced by I and $s - e + 1$, then (i), (ii), (iv) are equivalent according to [47, 5.3], whereas (ii) and (iii) are equivalent by [48, 4.1] and [28, 4.7]. Furthermore by Theorem 3.5(a.i), condition (i) holds for E if and only if the corresponding statement holds for I.

To show that (iii) for E implies (iii) for I, assume the former and let $\mathfrak{p}''$ be a prime of R'' with $\dim R''_{\mathfrak{p}''} = \ell(I_{\mathfrak{p}''}) = s - e + 1$. Write $\mathfrak{p} = \mathfrak{p}'' \cap R$. If $\dim R_{\mathfrak{p}} < s - e + 1$ then $E_{\mathfrak{p}}$ satisfies G_∞. Hence $I_{\mathfrak{p}''}$ has the same property by Proposition 3.2(b), and therefore $r(I_{\mathfrak{p}''}) = 0$. If on the other hand $\dim R_{\mathfrak{p}} = s - e + 1$ then $\mathfrak{p}'' = \mathfrak{p}R''$. Thus according to Proposition 3.10, $\ell(I_{\mathfrak{p}''}) = \ell(E_{\mathfrak{p}}) - e + 1$ and $r(I_{\mathfrak{p}''}) \leq r(E_{\mathfrak{p}})$. Hence $\dim R_{\mathfrak{p}} = \ell(E_{\mathfrak{p}}) - e + 1 = s - e + 1$ and we obtain $r(I_{\mathfrak{p}''}) \leq r(E_{\mathfrak{p}}) \leq s - e$.

It remains to prove that (v) for E is equivalent to the corresponding statement for I. So write $E = \sum_{i=1}^{n} Ra_i$. Let z_{ij}, $1 \leq i \leq n$, $1 \leq j \leq s+1$, be variables and

$$\widetilde{R} = R(\{z_{ij}\}), \quad \widetilde{E} = \widetilde{R} \otimes_R E, \quad x_j = \sum_{i=1}^{n} z_{ij} a_i \in \widetilde{E}.$$

Now (v) holds for E if and only if $F_0(\widetilde{E}/(x_1, \ldots, x_{s-1}, x_{s+1})) \subset F_0(\widetilde{E}/(x_1, \ldots, x_s))$, as can be seen by the same arguments as in [47, the proof of 3.8]. Since $e - 1 \leq s - 1$ it now follows that E satisfies (iv) if and only if $E''/(x_1, \ldots, x_{e-1}) \cong I$ does.

(b): Condition (a.i) and Theorem 3.5(a.i),(b) imply that $\mathcal{R}(I)$ is Cohen–Macaulay with $\mathcal{R}(I) \cong \mathcal{R}(E'')/(x_1, \ldots, x_{e-1})$. Now by [47, 5.3], I is of linear type, or else, $\ell(I) = s - e + 1$ and $\mathrm{r}(I) = s - e$. From Theorem 3.5(a.iii),(c.i,iii) it then follows that E is of linear type, or $\ell(E) = s$ and $\mathrm{r}(E) = s - e$.

It remains to show that U is a reduction of E with $\mathrm{r}_U(E) \leq s - e$.

Since E satisfies G_{s-e+1}, it follows that height $F_s(E) \geq s - e + 1$, where $s - e + 1 \geq 2$ by Proposition 4.1(a). Thus by (a.v), $\mathrm{Supp}(E/U) = V(F_0(E/U)) = V(F_s(E))$ has codimension $\geq s - e + 1 \geq 2$ in $\mathrm{Spec}(R)$. Now Proposition 3.2(a) shows that there exists a generic Bourbaki ideal of E with respect to U. Write N for this ideal and J for the image of U in N. By Proposition 3.2(b), (c) we may assume that

N is a perfect ideal of grade 2 satisfying G_{s-e+1}. Obviously, $F_{s-e+1}(N) = F_0(N/J)$. Thus by [39, 3.7], J is a reduction of N with $\mathrm{r}_J(N) \leq s-e$.

Finally, one always has the inclusion

$$F_0(E/U)\cdot E \subset F_s(E)\cdot U,$$

which by (a.v) implies

$$F_0(E/U)\cdot E = F_0(E/U)\cdot U.$$

Thus E is integral over U, and we may use (a.i) and Theorem 3.5(b),(c.ii) to deduce that $\mathrm{r}_U(E) = \mathrm{r}_J(N) \leq s-e$. □

Remark 4.8 If R contains an infinite field k, then one can assume that the row operations and the change of generators in Theorem 4.7(a.iv,v) are induced by an element of $\mathsf{Gl}_n(k)$.

Notice that Theorem 4.7 can be applied without prior knowledge of the analytic spread of E. Nevertheless the most interesting situation is where $s = \ell(E)$ or $s = \dim R + e - 1$.

Example 4.9 Even the most harmless modules of rank 2 may have a non–Cohen–Macaulay Rees algebra: Let R be a Gorenstein local ring of dimension 2 with infinite residue field, let $\mathfrak{a},\mathfrak{b}$ be complete intersection R–ideals of grade 2, and write $E = \mathfrak{a}\oplus\mathfrak{b}$. Then $\mathcal{R}(E)$ is Cohen–Macaulay if and only if $\mathfrak{a} = (a, yb)$ and $\mathfrak{b} = (xa, b)$ for some elements a,b,x,y of R, as can be easily seen from Theorem 4.7(a) (together with Propositions 2.3 and 4.1(a)).

Proposition 4.10 *Assume that one of the equivalent conditions of* Theorem 4.7(a) *hold with* $s = \ell(E)$*, and write* $\ell = \ell(E)$*,* $n = \nu(E)$*. Then the canonical module of* $\mathcal{R}(E)$ *is given as*

$$\omega_{\mathcal{R}(E)} \cong F_\ell(E)\mathcal{R}(E)(-e);$$

in particular $\mathcal{R}(E)$ *has type* $\binom{n-e}{n-\ell}$.

Proof. In addition to using the notation of the previous proof we write

$$\widetilde{I} = \widetilde{R}I \qquad \text{and} \qquad L = \mathrm{ann}(\widetilde{I}/(x_e,\ldots,x_\ell)).$$

As in that proof, $\widetilde{I}$ satisfies the same assumptions as E, and therefore $\omega_{\mathcal{R}(\widetilde{I})} \cong L\mathcal{R}(\widetilde{I})(-1)$ by [47, 2.7.a]. On the other hand, by [4] and the previous proof,

$$L = \mathrm{ann}(\widetilde{E}/(x_1,\ldots,x_\ell)) = F_0(\widetilde{E}/(x_1,\ldots,x_\ell)) = F_\ell(\widetilde{E}) = \widetilde{R}F_\ell(E).$$

Thus $\omega_{\mathcal{R}(\widetilde{I})} \cong F_\ell(E)\mathcal{R}(\widetilde{I})(-1)$.

Theorem 3.5(b) shows that $\mathcal{R}(\widetilde{I}) \cong \mathcal{R}(\widetilde{E})/(x_1,\ldots,x_{e-1})$, with $x_1,\ldots,x_{e-1}$ forming a regular sequence on the Cohen–Macaulay algebra $\mathcal{R}(\widetilde{E})$. Therefore

$$\omega_{\mathcal{R}(\widetilde{E})} \otimes \mathcal{R}(\widetilde{I}) \cong \omega_{\mathcal{R}(\widetilde{I})}(-e+1),$$

which implies that $\omega_{\mathcal{R}(\widetilde{E})}$ is generated by

$$[\omega_{\mathcal{R}(\widetilde{E})}]_e \cong [\omega_{\mathcal{R}(\widetilde{I})}]_1 \cong \widetilde{R}F_\ell(E).$$

This yields a natural homogeneous epimorphism

$$F_\ell(E) \otimes_R \mathcal{R}(\widetilde{E})(-e) \longrightarrow \omega_{\mathcal{R}(\widetilde{E})}.$$

As the graded components of both modules have the same ranks over $\widetilde{R}$ and the latter module is $\widetilde{R}$–torsionfree, one obtains the isomorphism $F_\ell(E)\mathcal{R}(\widetilde{E})(-e) \cong \omega_{\mathcal{R}(\widetilde{E})}$ and then

$$F_\ell(E)\mathcal{R}(E)(-e) \cong \omega_{\mathcal{R}(E)}.$$

Finally, since $L = F_0(\widetilde{E}/(x_1,\ldots,x_e))$ has height $\geq \ell - e + 1$ by the previous proof, it follows that L is the ideal of maximal minors of an $n-\ell$ by $n-e$ matrix having generic grade. Thus $\nu(L) = \binom{n-e}{n-\ell}$. On the other hand, $L = \widetilde{R}F_\ell(E)$, which gives $\text{type}(\mathcal{R}(E)) = \nu(\omega_{\mathcal{R}(E)}) = \binom{n-e}{n-\ell}$. □

We now describe the defining equations of one of these algebras.

Proposition 4.11 *Let $R = k[Y_1,\ldots,Y_d]$ be a polynomial ring in $d \geq 2$ variables over a field and let E be a finitely generated R–module with* proj dim $E = 1$ *that satisfies G_d. Assume that E has a presentation matrix φ whose entries are linear forms. Write $n = \nu(E)$, $\mathbf{Y} = Y_1,\ldots,Y_d$, $\mathbf{T} = T_1,\ldots,T_n$ for a new set of variables, and $B(\varphi)$ for the matrix whose entries are linear forms in $k[T_1,\ldots,T_n]$ satisfying $\mathbf{T}\cdot\varphi = \mathbf{Y}\cdot B(\varphi)$. Then $\mathcal{R}(E)$ is Cohen–Macaulay and $\mathcal{R}(E) \cong R[T_1,\ldots,T_n]/(\mathbf{Y}\cdot B(\varphi), I_d(B(\varphi)))$.*

Proof. By Proposition 3.2(a), there exists a generic Bourbaki ideal $I = I(E)$, $I \cong E''/(x_1,\ldots,x_{e-1})$. We may assume that I has a linear presentation matrix and is a perfect ideal of grade 2 satisfying G_d, by Proposition 3.2(b),(c). Now for I, our assertions follow from [38, 1.3] (if $\nu(I) > d$ and from [14, 9.1] if $\nu(I) \leq d$). Thus by Theorem 3.5(a.i),(b) $\mathcal{R}(E)$ is Cohen–Macaulay and $\mathcal{R}(I) \cong \mathcal{R}(E'')/(x_1,\ldots,x_{e-1})$ with $x_1,\ldots,x_{e-1}$ forming a regular sequence on $\mathcal{R}(E'')$. Taking a presentation $\mathcal{R}(E) \cong R[T_1,\ldots,T_n]/Q$, one easily sees that $(\mathbf{Y}\cdot B(\varphi), I_d(B(\varphi))) \subset Q$. Since in $R''[T_1,\ldots,T_n]$, $Q \subset (\mathbf{Y}\cdot B(\varphi), I_d(B(\varphi)), x_1,\ldots,x_{e-1})$, and the $x_1,\ldots,x_{e-1}$ are a regular sequence modulo $R''Q$, we deduce the desired equality $(\mathbf{Y}\cdot B(\varphi), I_d(B(\varphi))) = Q$. □

Projective dimension two

Here we will consider finitely generated modules over a Noetherian local ring that have a minimal free resolution of the form

$$0 \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow E \to 0,$$

with $b_i = \operatorname{rank} F_i$. Notice that if E is torsionfree, then $3 \leq \operatorname{grade} I_{b_2}(\varphi_2) \leq b_1 - b_2 + 1$. We will focus on the cases where these bounds are attained.

The first proposition generalizes a result of [49], where the module E was assumed to satisfy G_∞.

Proposition 4.12 *Let R be a Cohen–Macaulay local ring of dimension d with infinite residue field and let E be a finitely generated R–module with* $\operatorname{proj\,dim} E = 2$ *and* $b_2 = 1$. *Assume that $J = I_1(\varphi_2)$ is a strongly Cohen–Macaulay ideal of grade* 3 *and that E satisfies G_d and $r(E) \leq d-2$. Then $\mathcal{R}(E)$ is Cohen–Macaulay. Furthermore, E is of linear type, or else, $\ell(E) = d + e - 1$.*

Proof. First notice that E is torsionfree. By Proposition 3.2, E admits a generic Bourbaki ideal $I = I(E) \subset R''$ that has grade ≥ 2 and satisfies G_d. Furthermore $r(I) \leq r(E) \leq d-2$ by Proposition 3.10(b). Hence our assumptions descend from E to I. Since by Theorem 3.5(a.i,iii) and Proposition 3.10(a), all the asserted properties pass from I back to E, we may then replace R and E by R'' and I to assume that $E = I$ has a minimal free R–resolution of the form

$$0 \to R \longrightarrow R^n \longrightarrow R^n \longrightarrow I \to 0.$$

Write $Z_1(I)$ and $Z_1(J)$ for the modules of cycles in the Koszul complex of n generators of I and J, respectively. One has that $Z_1(I)^* \cong Z_1(J)$ and thus

$$Z_i(I) \cong (\wedge^i Z_1(I))^{**} \cong (\wedge^i Z_1(J))^* \cong (\wedge^{n-1-i} Z_1(J))^{**} \cong Z_{n-1-i}(J)$$

for every i (see [49, the proof of 2.2]). Since J is strongly Cohen–Macaulay of grade 3 it follows that depth $Z_\bullet(J) \geq d-1$, hence depth $Z_\bullet(I) \geq d-1$.

Now let $B_\bullet$, $Z_\bullet$, $H_\bullet$ denote the module of boundaries, cycles, homology in the Koszul complex of n generators of I. Since depth $Z_\bullet \geq d-1$, one obtains depth $H_\bullet \geq d-3$. Furthermore the exact sequence

$$0 \to B_{n-2} \longrightarrow Z_{n-2} \longrightarrow H_{n-2} \to 0$$

with $\operatorname{proj\,dim} B_{n-2} = 1$ implies that depth $H_{n-2} \geq d-2$. Thus I has the "sliding depth" property, depth $H_i \geq d - n + i$ for every i (see [17]). Since in addition, I satisfies G_d we conclude that either $\ell(I) = d$, or else I is of linear type and $\mathcal{R}(I)$ is Cohen–Macaulay ([48, 4.1]). Furthermore, I satisfies AN_{d-1} by [17, 3.3].

To establish the Cohen–Macaulayness of $\mathcal{R}(I)$ we may assume that $\ell(I) = d$. Write $S = \mathcal{S}(R^n)$ and consider the complex $\mathcal{Z}_\bullet$ of [14], which is a homogeneous complex of S–modules with $\mathcal{Z}_i = Z_i \otimes_R S(-i)$ and $H_0(\mathcal{Z}_\bullet) \cong S(I)$. Since I has sliding depth and is G_d, one knows from [14] that locally on the punctured spectrum of R, $\mathcal{Z}_\bullet$ is acyclic and I is of linear type. Thus, as depth $Z_\bullet \geq d-1$, the Acyclicity

Lemma shows that the graded components $\mathcal{Z}_{j\bullet}$ are acyclic whenever $j \leq d-1$ and then $S_j(I)$ is R–torsionfree whenever $j \leq d-2$. Now for every $1 \leq j \leq d-2$, $I^j \cong S_j(I)$ has depth $\geq d-1-j$, and so depth $R/I^j \geq d-2-j$. As $r(I) \leq d-2$ and I satisfies AN_2, we may use [28, 3.4] to conclude that $\mathcal{R}(I)$ is indeed Cohen–Macaulay. □

The next result provides a recipe for constructing modules with small reduction number.

Proposition 4.13 *Let R be a Gorenstein local ring of dimension 3 with infinite residue field and let U be a finitely generated torsionfree R–module with* proj dim $U = 2$ *and* $b_2 = 1$. *Assume that $I_1(\varphi_1) \subset I_1(\varphi_2)^2$ and that U satisfies G_∞. Write $J = I_1(\varphi_2)$, $V = U \otimes_R \operatorname{Quot}(R)$, and $E = U :_V J$. Then U is a minimal reduction of E with $\mathrm{r}_U(E) = \mathrm{r}(E) = 2$, and $\mathcal{R}(E)$ is Cohen–Macaulay.*

Proof. Since U is torsionfree one has grade $J \geq 3$, and from the G_∞ assumption it follows that $b_1 \leq 3$. Thus J is generated by a regular sequence x_1, x_2, x_3 and $b_1 = 3$. Write $e = \operatorname{rank} U = \operatorname{rank} E$, $n-1 = \nu(U)$, and notice that $n = e+3$.

There exists a free R–module G of rank e with $E \subset G \subset V$. Write $M = G/U$. Obviously $E = U :_G J$ and $E/U = 0 :_M J$. The minimal free R–resolution of the submodule $U \subset G$ gives a free resolution of M,

$$\mathbb{F}_\bullet : \qquad 0 \to R \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R^{n-1} \xrightarrow{\varphi_0} G \longrightarrow M \to 0.$$

With $\mathfrak{m}$ denoting the maximal ideal of R, we now claim that

$$E/U = H^0_{\mathfrak{m}}(M) \cong R/J. \tag{3}$$

It suffices to show this after completing R. Let $(\cdot)^\vee$ denote Matlis duals. From the resolution $\mathbb{F}_\bullet$ one sees that

$$H^0_{\mathfrak{m}}(M) \cong \operatorname{Ext}^3_R(M,R)^\vee \cong (R/J)^\vee \cong R/J.$$

Thus the second isomorphism of (3) holds. Furthermore $J \cdot H^0_{\mathfrak{m}}(M) = 0$, hence $H^0_{\mathfrak{m}}(M) \subset 0 :_M J$. Since on the other hand, $0 :_M J \subset H^0_{\mathfrak{m}}(M)$ and $0 :_M J = E/U$, the first equality in (3) now follows as well.

Formula (3) shows that E has finite projective dimension and $G/E \cong M/H^0_{\mathfrak{m}}(M)$ has depth > 0. Thus proj dim $E \leq 1$. It also implies that the Koszul complex $(\mathbb{K}_\bullet, \partial)$ on $\mathbf{x}$ resolves E/U. To obtain a resolution of G/E, let $u_\bullet : \mathbb{K}_\bullet \to \mathbb{F}_\bullet$ be a morphism of complexes lifting the inclusion map $E/U \hookrightarrow G/U = M$,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\partial_3} & R^3 & \xrightarrow{\partial_2} & R^3 & \xrightarrow{\partial_1} & R \\
 & & \downarrow u_3 & & \downarrow u_2 & & \downarrow u_1 & & \downarrow u_0 \\
0 & \longrightarrow & R & \xrightarrow{\varphi_2} & R^3 & \xrightarrow{\varphi_1} & R^{n-1} & \xrightarrow{\varphi_0} & G
\end{array}.$$

Notice that the entries of the product matrix

$$u_1\partial_2 = \varphi_1 u_2 \tag{4}$$

all lie in $J^2 = (\mathbf{x})^2$, by our assumption on φ_1. But ∂_2 is a differential in the Koszul complex on $\mathbf{x}$, and $\mathbf{x}$ form a regular sequence. Thus (4) shows that all the entries of u_1 are in $J = (\mathbf{x})$ (see also [30, the proof of 2.3]).

The mapping cone of $u_\bullet$ yields a free R–resolution of the factor module G/E,

$$C(u_\bullet): \qquad \cdots \longrightarrow R^3 \oplus R^3 \xrightarrow{\Phi} R \oplus R^{n-1} \xrightarrow{(u_0|\varphi_0)} G,$$

and shows that the submodule $E \subset G$ is presented by the n by 6 matrix

$$\Phi = \left(\begin{array}{ccc|ccc} -x_1 & -x_2 & -x_3 & 0 & 0 & 0 \\ \hline & & & & & \\ & u_1 & & & \varphi_1 & \\ & & & & & \end{array} \right).$$

Since $I_1(u_1) + I_1(\varphi_1) \subset J = (x_1, x_2, x_3)$ it follows that $I_1(\Phi)$ is generated by a single row of Φ. Furthermore, $\nu(E) = n$. We can now apply Theorem 4.7 with $s = n-1$. Since $\operatorname{proj\,dim} E \le 1$ and E satisfies G_3 where $3 = (n-1) - e + 1$, and $\nu(E) > n-1$, the theorem shows that $\ell(E) = n-1 = \nu(U)$, $\mathrm{r}_U(E) = \mathrm{r}(E) = (n-1) - e = 2$ and that $\mathcal{R}(E)$ is Cohen–Macaulay. □

Remark 4.14 If, in the setting of Proposition 4.13, G is *any* free R–module containing U, then necessarily $U :_G J = E$.

The following theorem has been proved in [43] under the additional assumption that $b_2 = 1$.

Theorem 4.15 *Let R be a Gorenstein local ring and let E be a finitely generated torsionfree R–module with* $\operatorname{proj\,dim} E = 2$ *that satisfies G_∞. Write $r = \nu(E) - \operatorname{rank} E$ $(= b_1 - b_2)$ and assume that* $\operatorname{proj\,dim} E_\mathfrak{p} \le 1$ *whenever* $\dim R_\mathfrak{p} \le r$ *(equivalently, $I_{b_2}(\varphi_2)$ has generic grade).*

(a) *If r is even then $\mathcal{R}(E)$ is Cohen–Macaulay and E is of linear type. But $\mathcal{R}(E)$ is not Gorenstein (in case $(r-2)!$ is a unit of R).*

(b) *If r is odd then $\mathcal{R}(E)$ is Gorenstein, but E is not of linear type.*

Proof. Consider a minimal free R–resolution of E,

$$0 \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow E \to 0.$$

Dualizing into R we obtain an exact sequence

$$0 \to B \longrightarrow F_1^* \xrightarrow{\varphi_2^*} F_2^* \longrightarrow N \to 0,$$

where

$$B = \ker \varphi_2^* \text{ and } N \cong \operatorname{Ext}_R^2(E,R).$$

Now since E is torsionfree, grade $\operatorname{Ext}_R^2(E,R) \geq 3$, and thus $\operatorname{Ext}_R^i(N,R) = 0$ for $i \leq 2$. Consequently,

$$0 \to F_2^{**} \xrightarrow{\varphi_2^{**}} F_1^{**} \longrightarrow B^* \to 0$$

is exact, which allows us to identify B^* with im $\varphi_1 \subset F_0$.

Write $S = \mathcal{S}(F_0)$ and let $\Psi : B^* \otimes_R S(-1) \to S$ be the homogeneous S–linear map induced by the inclusion $B^* = \text{im } \varphi_1 \subset F_0 = [S]_1$. Write $I(\Psi) = \text{im } \Psi \subset S$ and $J(\Psi) = I(\Psi)^{\text{unm}}$, which is the intersection of the primary components of $I(\Psi)$ having maximal dimension. As $B^* = \text{im } \varphi_1$, one has $S/I(\Psi) \cong \mathcal{S}(E)$, and since E satisfies G_∞ one concludes that height $I(\Psi) = r$ and $S/J(\Psi) \cong \mathcal{R}(E)$ ([51]).

The ideals $I(\Psi)$ and $J(\Psi)$ however were studied in [37] and [35] (notice that $r \geq 2$ since proj dim $E > 1$; in [37] the modules are assumed to be graded over a homogeneous ring, but this assumption can be deleted). Now $I(\Psi) = J(\Psi)$ (i.e. E is of linear type) if and only if r is even ([37, 4.5.a]). Furthermore $S/J(\Psi)$ is Cohen–Macaulay in any case and is Gorenstein if r is odd ([37, 4.7.c,d]). Finally, if r is even and $(r-2)!$ is a unit, then $S/J(\Psi) = S/I(\Psi)$ cannot be Gorenstein ([35, 2.3,4.3], cf. also [43] for the case $b_2 = 1$). □

Remark 4.16 The assumption that proj dim $E_{\mathfrak{p}} \leq 1$ if $\dim R_{\mathfrak{p}} \leq r$, is automatically satisfied if E has projective dimension ≤ 1 locally on the punctured spectrum.

Example 4.17 Let R be a regular local ring of dimension 4 and let E be a reflexive vector bundle with $\nu(E) \leq \text{rank } E + 3$. Then $\mathcal{R}$ is Cohen–Macaulay. The same holds for $E = \mathfrak{p}$ a four generated prime ideal that is a complete intersection locally on the punctured spectrum.

The next corollary deals with the "Factorial Conjecture" ([15, 7.2], [44, 2.8]).

Corollary 4.18 *Let R be a Gorenstein local ring of dimension d in which $(d-3)!$ is a unit and let E be a finitely generated R–module with* proj dim $E = 2$. *Then E cannot be of linear type if $\mathcal{R}(E)$ is quasi–Gorenstein.*

Proof. We may assume that E is torsionfree and satisfies G_∞ since otherwise it cannot be of linear type. After localizing we may further suppose that E has projective dimension ≤ 1 locally on the punctured spectrum. But then by Theorem 4.15, E is either not of linear type, or else, $\mathcal{R}(E)$ is Cohen–Macaulay and not Gorenstein. □

Corollary 4.19 *Let R be a Gorenstein local ring of dimension d in which $(d-3)!$ is a unit and let I be an unmixed ideal of grade* 2 *with* proj dim $I \leq 2$. *If I is of linear type and normally torsionfree then I is perfect.*

Proof. After localizing we may assume that I has projective dimension ≤ 1 locally on the punctured spectrum. But then $\mathcal{R}(I)$ is Cohen–Macaulay by Theorem 4.15. In the light of Corollary 4.18, it suffices to check that $\mathcal{R}(I)$ is Gorenstein.

Indeed, since I is unmixed, generically a complete intersection, and normally torsionfree, it follows that the Cohen–Macaulay ring $G = \mathrm{gr}_I(R)$ is Gorenstein ([16, the proof of 1.1], [29, the proof of 3.2]). As moreover grade $I = 2$, one has $a(G) = -2$. Thus by [26, 3.1], $\mathcal{R}(I)$ is Gorenstein as well. □

5 Ideal Modules

In this section we consider a class of modules that, like ideals, afford a *natural* embedding into a free module of the same rank. They provide the notions of (almost) complete intersection module, equimultiple module, link via a module, in analogy to the case of ideals.

Basic properties

Let R be a Noetherian ring and E an R–module. We say that E is an *ideal module* if $E \neq 0$ is finitely generated and torsionfree and the double dual E^{**} is free. There are various ways of characterizing such modules.

Proposition 5.1 *Let R be a Noetherian ring and $E \neq 0$ an R–module. The following are equivalent:*

(a) *E is an ideal module.*

(b) *E is finitely generated and torsionfree and E^* is free.*

(c) *$E \subset G$ where G is a free R–module of finite rank and* grade $G/E \geq 2$.

(d) *$E \cong \mathrm{im}\,\varphi$ where φ is a homomorphism of free R–modules of finite rank and* grade $\mathrm{coker}(\varphi) \geq 2$.

(e) *$E \cong \mathrm{im}\,\psi^*$ where*
$$0 \to G \xrightarrow{\psi} F \longrightarrow M \to 0$$
is a finite free resolution of a torsionfree R–module M.

Proof. (a) $\Leftrightarrow$ (b): See, e.g., [3, 1.4.21.b].

(b) $\Rightarrow$ (e): Consider an exact sequence
$$0 \to U \longrightarrow F^* \longrightarrow E \to 0,$$
where F is a free R–module of finite rank. Dualizing, one obtains
$$0 \to E^* \xrightarrow{\psi} F \longrightarrow U^*,$$

where E^* is free of finite rank. Since E has a rank (see, e.g., [3, 1.4.22]), it follows that $E \cong \operatorname{im} \psi^*$.

(e) $\Rightarrow$ (d): One takes φ to be ψ^*. Notice that coker $\varphi = \operatorname{Ext}^1_R(M,R)$, which has grade ≥ 2 since M is a torsionfree module of finite projective dimension.

(d) $\Rightarrow$ (c): This is clear.

(c) $\Rightarrow$ (b): Dualizing the exact sequence

$$0 \to E \longrightarrow G \longrightarrow G/E \to 0$$

one sees that $E^* \cong G^*$ is free. $\square$

Examples of ideal modules are finite direct sums of ideals of grade ≥ 2, as well as finitely generated torsionfree $\neq 0$ modules over regular local rings of dimension 2. Furthermore by the above proposition, the last map in any finite free resolution of length ≥ 2 gives rise to an ideal module. Concrete examples are given by the module whose symmetric algebra defines the ideal of the variety of commuting matrices (cf. [51, Chapter 9] and by the jacobian module of a normal complete intersection $R = k[\mathbf{X}]/\mathfrak{a}$ (the jacobian module is the image of the R-linear map represented by the jacobian matrix of a set of generators of $\mathfrak{a}$).

The above proposition also shows that if E is an ideal module, then E has a rank $e > 0$, $E_{\mathfrak{p}}$ is free whenever depth $R_{\mathfrak{p}} \leq 1$, and E is orientable. Furthermore, any inclusion $E \subset G$ as in part (b) is the natural embedding $E \hookrightarrow E^{**}$ followed by an isomorphism. Since an ideal module is free if and only it is reflexive, we conclude that the nonfree locus $V(F_e(E))$ of E coincides with $\operatorname{Supp}(G/E) = V(F_0(G/E)) = V(I_e(\varphi))$.

Recall that a module over a local ring is said to be a *vector bundle* if it is free locally on the punctured spectrum.

Proposition 5.2 *Let R be a Noetherian local ring of dimension d, let E be an ideal module that is not free, and write e for the rank of E and c for the codimension of its nonfree locus* $(=$ height $F_e(E) =$ height $F_0(G/E) =$ height $I_e(\varphi))$. *Then $\ell(E) \geq c + e - 1$. In particular if E is a vector bundle then $\ell(E) = d + e - 1$.*

Proof. We may assume that the residue field of R is infinite. Let U be a minimal reduction of E and write $n = \nu(U)$. Since $U^{**} = E^{**}$, U is again an ideal module of rank e whose nonfree locus coincides with the nonfree locus of E. Thus we may replace E by U to assume that $\ell(E) = n$. Now as φ is an e by n matrix and $I_e(\varphi) \neq R$, Macaulay's bound yields height $I_e(\varphi) \leq n - e + 1$, or equivalently, $\ell(E) \geq c + e - 1$. The second assertion of the proposition now follows as well, because $\ell(E) \leq d + e - 1$ by Proposition 2.3. $\square$

Let R be a Noetherian local ring and E an ideal module of rank e. As in the case of ideals, we define the *deviation* of E as $d(E) = \nu(E) - e + 1 - \operatorname{grade} F_e(E)$ and its *analytic deviation* as $ad(E) = \ell(E) - e + 1 - \operatorname{grade} F_e(E)$. Notice that if E is not free then $d(E) \geq ad(E) \geq 0$ by Proposition 5.2.

Accordingly, we say that E is a *complete intersection module*, an *almost complete intersection module*, or an *equimultiple module* if E is an ideal module and if $d(E) \leq 0$, $d(E) \leq 1$, or $ad(E) \leq 0$, respectively. These definitions coincide with the corresponding notions for ideals provided the ideals are proper and have grade ≥ 2.

Bourbaki ideals of ideal modules

(Generic) Bourbaki ideals of ideal modules can be described rather explicitly: Let R be a Noetherian local ring and E an ideal module of rank e. Let φ be a matrix with e rows whose columns generate the image of E in $G = E^{**} \cong R^e$ (see Proposition 5.1). By Proposition 3.2(a) there exists a generic Bourbaki ideal $N \cong G''/F$ of G with respect to E. Let Φ be an e by $e-1$ matrix whose columns generate $F \subset G''$, and let $\Delta_1, \ldots, \Delta_e$ denote the signed maximal minors of Φ. Notice that N is generated by $\Delta_1, \ldots, \Delta_e$ and that the image of E''/F in N is a generic Bourbaki ideal $I(E)$ of E (Remark 3.4(b)). Thus $I(E)$ is generated by the entries of the product matrix

$$[\Delta_1, \ldots, \Delta_e] \cdot \varphi,$$

which, after elementary column operations on φ over R'', consist of the e by e minors of φ fixing the first $e-1$ columns.

If $\mathfrak{a}_i$ are R–ideals of grade ≥ 2 and

$$E = \bigoplus_{i=1}^{e} \mathfrak{a}_i \subset G = R^e,$$

then

$$I(E) = \sum_{i=1}^{e} \Delta_i \mathfrak{a}_i,$$

where now Φ is a matrix whose i–th row consists of $e-1$ "generic" elements of $\mathfrak{a}_i$ (for $e = 2$ and grade $\mathfrak{a}_i = 2$, this construction has been known as "liaison addition", see [41]). The above formula simplifies to $I(E) = I_{e-1}(\Phi) \cdot \mathfrak{a}$ in case $\mathfrak{a}_1 = \cdots = \mathfrak{a}_e = \mathfrak{a}$. Curiously, $I_{e-1}(\Phi) \cdot \mathfrak{a}$ has a Cohen–Macaulay Rees algebra if $\mathcal{R}(\mathfrak{a})$ is Cohen–Macaulay — this follows from Theorem 3.5(a.i), since $E = \bigoplus_{i=1}^{e} \mathfrak{a}$ has a Cohen–Macaulay Rees algebra by [13, 2.5].

One of the difficulties we will encounter is that the deviation and analytic deviation of an ideal module tend to increase as one passes to a generic Bourbaki ideal (see Remark 3.2(c) and Proposition 3.10(a)). The following technical result will play a crucial role in dealing with this problem:

Proposition 5.3 *Let R be a Cohen–Macaulay local ring and E an ideal module. Let U be a submodule of E. Assume that* grade $E/U \geq s \geq 2$ *and that E is free locally in codimension $s-1$. Further let I be the generic Bourbaki ideal of E with respect to U (which exists by* Proposition 3.2(a)). *Then I satisfies G_s and AN_{s-1}.*

Proof. The first assertion follows from Proposition 3.2(b). To verify the Artin–Nagata property AN_{s-1}, write $I \cong E''/F$ and notice that

$$\text{grade } E^{**}/E \geq \text{grade } E^{**}/U \geq s \geq 2.$$

Thus by Proposition 3.2, there exists a generic Bourbaki ideal $N \cong E^{**\prime\prime}/F$ of E^{**} with respect to U that satisfies G_s and is either the unit ideal or else a perfect ideal of grade 2. In particular, N satisfies AN_{s-1} ([23, 3.1]).

On the other hand, I differs from the image of E''/F in N only by multiplication with a unit of $\text{Quot}(R'')$, which does not affect the Artin–Nagata property. Thus we may assume that I is the image of E''/F in $E^{**\prime\prime}/F$. But then grade $N/I =$ grade $E^{**}/E \geq s$, or equivalently, height $I:N \geq s$. Now by [46, 1.12] the property AN_{s-1} descends from N back to I. □

Analytic deviation at most one

Theorem 5.4 *Let R be a Cohen–Macaulay local ring of dimension d and let E be a finitely generated torsionfree R–module that has a rank e and is free locally in codimension one. Assume that E is integral over an almost complete intersection module with $\nu(U) = s$ and that E satisfies G_{s-e+1}.*

(a) *If $E = U$ then E is of linear type.*

(b) *If $\mathrm{r}_{U_\mathfrak{p}}(E_\mathfrak{p}) \leq 1$ for every prime $\mathfrak{p}$ with $\dim R_\mathfrak{p} = \ell(E_\mathfrak{p}) - e + 1 = s - e + 1$, then $\mathcal{R}(E)$ is Cohen–Macaulay if and only if* depth $E \geq d - s + e$. *In this case either E is of linear type, or else, U is a minimal reduction of E and $\mathrm{r}_U(E) = \mathrm{r}(E) = 1$ (if R has an infinite residue field).*

Proof. We write $G = U^{**}$ and notice that $U \subset E \subset G$. In particular, E is an ideal module and E is free locally in locally in codimension $s - e - 1$. Furthermore we may assume that $s > e$ because otherwise U is free, hence E is free.

As in the proof of Theorem 4.7(a), write $E = \sum_{i=1}^{n} Ra_i$, $\widetilde{R} = R(\{z_i\})$, $1 \leq i \leq n$, $1 \leq j \leq n$, $\widetilde{E} = \widetilde{R} \otimes_R E$, $x_j = \sum_{i=1}^{n} z_{ij} a_i \in \widetilde{E}$, and $F = \sum_{j=1}^{e-1} \widetilde{R} x_j$. We may replace R, E, U by $\widetilde{R}, \widetilde{E}, \sum_{j=1}^{s} \widetilde{R} x_j$ to assume that $F \subset U$ (and that the residue field of R is infinite), without changing our assumptions and conclusions (see [45, the proof of 3.4]). Now by Proposition 3.2(a),(c), E/F can be identified with a generic Bourbaki ideal $I = I(E)$ with grade $I \geq 2$.

Write J for the image of U in I, which yields a reduction of I with $\nu(J) = s - e + 1$ and $\mathrm{r}_J(I) \leq \mathrm{r}_U(E)$. Set $\ell = \ell(I)$ and notice that $\ell \leq s - e + 1$. Furthermore we know that I satisfies G_{s-e+1} by Proposition 3.2(b) and AN_{s-e-1} by Proposition 5.3.

(a): In this case $I = J$. But then I is generated by a d–sequence ([46, 1.8.b]) and hence I is of linear type ([20, 3.1]). Now Theorem 3.5(a.iii) implies that E is of linear type as well.

(b): First notice that if $\mathcal{R}(E)$ is Cohen–Macaulay, then $\mathcal{R}(I)$ is Cohen–Macaulay by Theorem 3.5(a.i). Thus $\mathrm{gr}_I(R)$ is Cohen–Macaulay, which gives depth $R/I \geq d - \ell(I) \geq d - (s - e + 1)$ ([10, 3.3] and Proposition 3.10(a)). Therefore depth $E \geq d - s + e$.

¿From now on we assume that depth $E \geq d-s+e$ and $\mathrm{r}_{U_\mathfrak{p}}(E_\mathfrak{p}) \leq 1$ whenever $\dim R_\mathfrak{p} = s-e+1$. We need to show that $\mathcal{R}(E)$ is Cohen–Macaulay, and that E is of linear type, or else, $\ell(E) = s$ and $\mathrm{r}_U(E) = \mathrm{r}(E) = 1$. By our assumption, depth $R/I \geq d-(s-e+1)$ since $s > e$.

We first treat the case where $\mathrm{r}(I) = 0$. In this case $\nu(I) \leq s-e+1$. Thus I satisfies G_∞. Since moreover I is AN_{s-e-1} and depth $R/I \geq d-(s-e+1)$ with $s-e+1 \geq \nu(I)$, it follows that I has the sliding depth property ([17, 3.4] or [46, 1.8.a]). This forces $\mathcal{R}(I)$ to be Cohen–Macaulay and I to be of linear type ([14, 9.1]). But then by Theorem 3.5(a.i,iii), $\mathcal{R}(E)$ is Cohen–Macaulay and E is of linear type.

Next we assume that $\mathrm{r}(I) \geq 1$. If $\ell = \ell(I) < s-e+1$, write L for a minimal reduction of I. Since I satisfies G_{s-e+1} and AN_{s-e-1}, one has height $L:I \geq s-e+1 > \ell = \nu(L)$ ([46, 1.11]) and therefore $I = L$ ([46, 1.7.a]), contradicting the assumption $\mathrm{r}(I) \geq 1$. Thus $\ell = \ell(I) = s-e+1$. In this case J is a minimal reduction of I and therefore $1 \leq \mathrm{r}(I) \leq \mathrm{r}_J(I)$. Now let $\mathfrak{p}''$ be a prime of R'' with $\dim R''_{\mathfrak{p}''} = \ell(I_{\mathfrak{p}''}) = \ell$ and write $\mathfrak{p} = \mathfrak{p}'' \cap R$. If $\dim R_\mathfrak{p} < \ell$ then $\dim R_\mathfrak{p} \leq s-e$. Hence $\mu(E_\mathfrak{p}) \leq s-1$ and so $E_\mathfrak{p}$ is a complete intersection. Thus $E_\mathfrak{p}$ is of linear type by part (a), which gives $\mathrm{r}_{J_{\mathfrak{p}''}}(I_{\mathfrak{p}''}) \leq \mathrm{r}_{U_\mathfrak{p}}(E_\mathfrak{p}) = 0$. If on the other hand $\dim R_\mathfrak{p} = \ell$ then $\mathfrak{p}'' = \mathfrak{p}R''$. Hence by Proposition 3.10(a), $\ell(I_{\mathfrak{p}''}) = \ell(E_\mathfrak{p}) - e + 1$ and therefore $\dim R_\mathfrak{p} = \ell(E_\mathfrak{p}) - e + 1 = s-e+1$. Now our assumption on E yields $\mathrm{r}_{J_{\mathfrak{p}''}}(I_{\mathfrak{p}''}) \leq \mathrm{r}_{U_\mathfrak{p}}(E_\mathfrak{p}) \leq 1$. We conclude that $\mathrm{r}_{J_{\mathfrak{p}''}}(I_{\mathfrak{p}''}) \leq 1$ whenever $\dim R''_{\mathfrak{p}''} = \ell(I_{\mathfrak{p}''}) = \ell$. Furthermore I satisfies G_ℓ and $AN_{\ell-2}$, and depth $R/I \geq d-\ell$. Hence $\mathrm{r}_J(I) \leq 1$ by [28, 4.7 and its proof], which gives $\mathrm{r}(I) = \mathrm{r}_J(I) = 1$. The Cohen-Macaulayness of $\mathcal{R}(I)$ now follows from [28, 3.4]. Thus $\mathcal{R}(E)$ is Cohen-Macaulay by Theorem 3.5(a.i). Furthermore according to Theorem 3.5(b), (c.i,ii), $\mathrm{r}(E) = \mathrm{r}(I) = 1$, $\mathrm{r}_U(E) = r_J(I) = 1$, and by Proposition 3.10, $\ell(E) = \ell(I) + e - 1 = s$. □

Corollary 5.5 *Let R be a Cohen–Macaulay local ring of dimension d and let E be an almost complete intersection module with $n = \nu(E)$ and $e = \operatorname{rank} E$ satisfying G_{n-e+1}.*

(a) *E is of linear type.*

(b) *$\mathcal{R}(E)$ is Cohen–Macaulay if and only* depth $E \geq d-n+e$.

Proof. One applies Theorem 5.3 with $U = E$ and $s = n$. □

Corollary 5.6 (see also [32, 3.2]) *Let R be a Cohen–Macaulay local ring of dimension d and E a complete intersection module. Then E is of linear type and $\mathcal{R}(E)$ is Cohen–Macaulay.*

Proof. Write $d = \dim R$, $n = \nu(E)$ and $e = \operatorname{rank} E$. Notice that E satisfies G_{n-e+1} and that depth $E \geq d-n+e$ by the Buchsbaum–Rim complex ([5]). Now the assertions follow from Corollary 5.5. □

Corollary 5.7 *Let R be a Cohen–Macaulay local ring of dimension d with infinite residue field and let E be an ideal module with $\ell = \ell(E)$ and $e = \operatorname{rank} E$ satisfying $G_{\ell-e+1}$. Assume that $ad(E) \leq 1$ and* $\mathrm{r}(E_\mathfrak{p}) \leq 1$ *for every prime $\mathfrak{p}$ with* $\dim R_\mathfrak{p} = \ell(E_\mathfrak{p}) - e + 1 = \ell - e + 1$. *Then $\mathcal{R}(E)$ is Cohen–Macaulay if and only if* depth $E \geq d-\ell+e$. *In this case either E is of linear type, or else,* $\mathrm{r}(E) = 1$.

Proof. It suffices to prove the assertion for the $\widetilde{R}$-module $\widetilde{E}$ as defined in the proof of Theorem 5.4. This module is integral over the almost complete intersection module $U = \sum_{j=1}^{\ell} \widetilde{R}x_j$. Let $\tilde{\mathfrak{p}}$ be any prime of $\widetilde{R}$ with $\dim \widetilde{R}_{\tilde{\mathfrak{p}}} = \ell(\widetilde{E}_{\tilde{\mathfrak{p}}}) - e + 1 = \ell - e + 1$, and write $\mathfrak{p} = \tilde{\mathfrak{p}} \cap R$. One has $\tilde{\mathfrak{p}} = \mathfrak{p}\widetilde{R}$ and therefore $\mathrm{r}_{U_{\tilde{\mathfrak{p}}}}(\widetilde{E}_{\tilde{\mathfrak{p}}}) = \mathrm{r}(E_{\mathfrak{p}}) \leq 1$ by [45, the proof of 3.4]. The corollary now follows from Theorem 5.4(b). $\square$

Corollary 5.8 *Let R be a Cohen–Macaulay local ring of dimension d with infinite residue field and let E be an ideal module of rank e. Assume that E is free locally in codimension $d-2$, satisfies G_d, and* $\mathrm{r}(E) \leq 1$. *Then $\mathcal{R}(E)$ is Cohen–Macaulay. Furthermore either E is of linear type, or else, $\ell(E) = d + e - 1$ and* $\mathrm{r}(E) = 1$.

Proof. Set $s = d + e - 1$. Now by Proposition 2.3, $\ell(E) \leq s$, and by Corollary 5.6 we may assume that $\nu(E) \geq s$. Thus there exists a reduction U of E with $\nu(U) = s$ and $\mathrm{r}_U(E) \leq \mathrm{r}(E) \leq 1$. Notice that U is free locally in codimension $d - 2 = s - e - 1$, hence U is an almost complete intersection module. Now again Theorem 5.4 implies the assertion. $\square$

In the next two subsections we describe methods for constructing modules of reduction number one that are reminiscent of [7] (see also [6]). The underlying philosophy is that socles tend to give rise to such modules. Our main result on the other hand is new even for the case of ideals.

Socles modulo ideals generated by d-sequences

Lemma 5.9 ([50, the proof of 5.4]) *Let R be a Cohen–Macaulay local ring and let $J \subset N$ be R–ideals with $\nu(J) \leq s \leq$* height $J : N$. *Assume that J satisfies G_∞ and sliding depth. Then for every prime ideal $\mathfrak{p}$ of height $> s$, $\mathfrak{p} \in \mathrm{Ass}(R/JN)$ if and only if $\mathfrak{p} \in \mathrm{Ass}(R/N)$.*

Proof. Let H_1 denote the first Koszul homology of a generating sequence $a_1, \ldots, a_s$ of J. By our assumption, H_1 is annihilated by N ([50, the proof of 4.12]). Furthermore the sequence

$$0 \to H_1 \longrightarrow \oplus R/J \longrightarrow J/J^2 \to 0$$

is exact. After tensoring with $\otimes_R R/N$, we obtain an exact sequence

$$0 \to H_1 \longrightarrow \oplus R/N \longrightarrow J/JN \to 0. \tag{5}$$

Since $\mathrm{depth}(H_1)_{\mathfrak{p}} \geq 2$ and $\mathrm{depth}(R/J)_{\mathfrak{p}} \geq 1$, (5) now shows that $\mathrm{depth}(R/N)_{\mathfrak{p}} = 0$ if and only if $\mathrm{depth}(R/JN)_{\mathfrak{p}} = 0$. $\square$

Our next theorem says that ideals defining the socle modulo certain d–sequences have reduction number one, but arbitrary analytic deviation.

Theorem 5.10 *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d with infinite residue field k and let J be an R–ideal satisfying G_∞ and sliding depth. Write N for the intersection of all primary components of J which are not $\mathfrak{m}$–primary* (i.e., $N/J = H^0_{\mathfrak{m}}(R/J)$) *and assume that $k \otimes_R J \not\to k \otimes_R N$. Further set $I = J : \mathfrak{m}$.*

(a) $\mathrm{r}(I) \leq 1$, $\mathrm{gr}_I(R)$ *is Cohen–Macaulay, and if $d \geq 2$ and* $\operatorname{grade} J > 0$, *then $\mathcal{R}(I)$ is Cohen–Macaulay.*

(b) (i) *If $\nu(I) \leq d$, then $\nu(I/J) = 1$ and I is of linear type.*

(ii) *If $\nu(I) > d$, then J is a reduction of I, $\ell(I) = d$, and* $\mathrm{r}_J(I) = \mathrm{r}(I) = 1$.

(c) *Case* (b.ii) *occurs if one of the following conditions holds:*

(i) $\nu(N) > d$;

(ii) $\nu(N/J) > 1$;

(iii) *R is normal, for every $\mathfrak{p} \in V(J)$ with $\mathfrak{p} \neq \mathfrak{m}$, either $\nu(J_{\mathfrak{p}}) < \dim R_{\mathfrak{p}}$ or $J_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$, and J is not a normal ideal.*

Proof. (Item (7) and (8) below were inspired by [6, the proof of 2.2]). Notice that depth $R/J = 0$ since $J \neq N$, and that $\nu(J) = d$ since J has sliding depth with depth $R/J = 0$. Furthermore J is of linear type ([14, 9.1]). Also $d > 0$ as $J \neq 0$. Finally $I \neq J$ and we may assume that $I \neq R$.

We are going to prove the following assertion:

If $\nu(I) > d$ or $\nu(I/J) > 1$ or if one of the conditions in (c) holds, then

$$I^2 = JI. \tag{6}$$

We first show that (6) implies the theorem.

Case 1: J is not a reduction of I. In this case $\nu(I) \leq d$ and $\nu(I/J) = 1$ by (6). But then since J satisfies G_∞, one can find a generating sequence $b_1, \ldots, b_d$ of I so that $(b_1, \ldots, b_{d-1}) \subset J$ and height $(b_1, \ldots, b_i) : J \geq i$, height $(J + [(b_1, \ldots, b_i) : J]) \geq i+1$ whenever $0 \leq i \leq d-1$ ([46, 1.4 and its proof]). By the sliding depth assumption, $R/(b_1, \ldots, b_i) : J$ is a Cohen–Macaulay ring and depth $R/(b_1, \ldots, b_i) \geq d-i > 0$ ([17, 3.3 and 3.7] or [46, 1.7.b]). Now since $\mathfrak{m}$ is not an associated prime of $(b_1, \ldots, b_i)$ and height $J : I = d$, it follows that $(b_1, \ldots, b_i) : I = (b_1, \ldots, b_i) : J$. Thus $R/(b_1, \ldots, b_i) : I$ is Cohen–Macaulay for $0 \leq i \leq d-1$, and therefore $I = (b_1, \ldots, b_d)$ has sliding depth ([17, 3.7] or [46, 1.8.c]). As I satisfies G_∞ and the sliding depth condition, it now follows that I is of linear type, $\mathrm{gr}_I(R)$ is Cohen–Macaulay and so is $\mathcal{R}(I)$ if grade $J > 0$ ([14, 9.1]). Recall that necessarily $\nu(I) \leq d$ and $\nu(I/J) = 1$.

Case 2: J is a reduction of I. Being of linear type, J is necessarily a minimal reduction of I. Consequently, $\nu(I) > \ell(I) = \nu(J) = d$ and $\mathrm{r}(I) \leq \mathrm{r}_J(I) \leq 1$ by (6). But then [50, 4.13] (or its modification in [46, 4.8]) implies the Cohen–Macaulayness of $\mathrm{gr}_I(R)$. Thus $\mathcal{R}(I)$ is Cohen–Macaulay as well if $d \geq 2$ and grade $J \geq 0$ ([45, 3.6]). Recall that $\nu(I) > d$ in this case.

Therefore it suffices indeed to prove (6). From now on we suppose that its assumptions are satisfied. Let $a_1,\dots,a_d$ be any generating sequence of J. Write $K_i=(a_1,\dots,\widetilde{a_i},\dots,a_d):J$, $\mathfrak{a}=(a_1,\dots,a_{d-1})$, $K=K_d$, and let $\overline{}$ denote images in $\overline{R}=R/K$. Recall that if height $K\geq d-1$ and height $J+K\geq d$, then $\overline{R}$ is a Cohen–Macaulay ring of dimension one, a_d is regular on $\overline{R}$, and $J\cap K=\mathfrak{a}$ ([17, 3.3]). In fact, $I\cap K=\mathfrak{a}$ and $N\cap K=\mathfrak{a}$, because $\mathfrak{m}$ is not an associated prime of $\mathfrak{a}$ ([17, 3.7] or [46, 1.7.b]) and height $J:I=$ height $J:N=d$. Furthermore since J satisfies AN_{d-2} ([17, 3.3]), $\mathfrak{a}$ does ([46, 1.12]) and therefore has sliding depth ([17, 3.4] or [46, 1.8.c]). Thus by Lemma 5.9, $\mathfrak{m}$ cannot be an associated prime of $\mathfrak{a}N$. As height $N+K\geq d$, we then conclude that $\mathfrak{a}\cap N^2=\mathfrak{a}N$.

Now we prove that $\overline{R}$ cannot be a discrete valuation ring (still assuming height $K\geq d-1$, height $J+K\geq d$). So suppose the contrary. As $I\cap K=\mathfrak{a}=N\cap K$, it follows that $I/\mathfrak{a}=\overline{I}\subset\overline{R}$ and $N/\mathfrak{a}=\overline{N}\subset\overline{R}$ are cyclic R–modules, which already contradicts each of the assumptions $\nu(I)>d$, $\nu(I/J)>1$, (c.i), (c.ii). As to (c.iii), notice that $\mathcal{R}(J)$ is Cohen–Macaulay. Furthermore, the deformation argument of [25, the proof of 3.2] shows that $\mathcal{R}(J)_{\mathfrak{m}\mathcal{R}(J)}$ is a discrete valuation ring if $\overline{R}$ has this property. But then with the assumptions of (c.iii), $\mathcal{R}(J)$ would be normal, which again yields a contradiction.

Next, we prove that

$$\mathfrak{m}I=\mathfrak{m}J. \tag{7}$$

If (7) does not hold then $\mathfrak{m}I$ contains elements of $J\setminus\mathfrak{m}J$. We may assume that $a_d\in\mathfrak{m}I$ and still height $K\geq d-1$, height $J+K\geq d$ ([46, 1.4]). But then $\overline{\mathfrak{m}}\overline{I}=(\overline{a_d})\cong\overline{R}$, which would force $\overline{R}$ to be a discrete valuation ring. This concludes the proof of (7).

Since $I^2\subset J$, every element of I^2 can be written in the form $\sum_{i=1}^{d}r_ia_i$ with $r_i\in R$. We now show that necessarily

$$r_i\in(K_i,a_i):\mathfrak{m}, \tag{8}$$

for every i with height $K_i\geq d-1$ and height $J+K_i\geq d$. It suffices to do this for $i=d$. Now indeed, multiplying the inclusion $\sum_{i=1}^{d}r_ia_i\in I^2$ by $\mathfrak{m}$, using the fact that $\mathfrak{m}I^2=\mathfrak{m}J^2\subset J$ by (7), and reducing modulo K, one sees that $\overline{\mathfrak{m}}\overline{r_d}\overline{a_d}\subset(\overline{a_d^2})$. Thus $\overline{r_d}\in(\overline{a_d^2}):(\overline{\mathfrak{m}}\overline{a_d})=(\overline{a_d}):\overline{\mathfrak{m}}$ since $\overline{a_d}$ is a non zerodivisor of $\overline{R}$. This yields $r_d\in(K,a_d):\mathfrak{m}$, showing (8).

Now we claim that

$$I^2\subset NJ. \tag{9}$$

Since $\nu(J)=d$ and $k\otimes_R J\not\to k\otimes_R N$, we may choose the generating sequence $a_1,\dots,a_d$ so that $a_d\in\mathfrak{m}N$ and height $K\geq d-1$, height $J+K\geq d$ ([46, 1.4]). By (8), $I^2\subset\mathfrak{a}+[(K,a_d):\mathfrak{m}]a_d$. However in $\overline{R}$,

$$(\overline{a_d}):\overline{\mathfrak{m}}=\overline{\mathfrak{m}}^{-1}\overline{a_d}\subset\overline{\mathfrak{m}}^{-1}\overline{\mathfrak{m}}\overline{N}\subset\overline{N},$$

thus $(K,a_d):\mathfrak{m}\subset N+K$. Since furthermore $\mathfrak{a}\cap N^2=\mathfrak{a}N$, we obtain

$$I^2\subset[\mathfrak{a}+(N+K)a_d]\cap N^2=(\mathfrak{a}+Na_d)\cap N^2=(\mathfrak{a}\cap N^2)+Na_d=N\mathfrak{a}+Na_d=NJ.$$

Now finally we verify that $I^2 \subset JI$. For this we change the generating sequence $a_1,\ldots,a_d$ to assume that height $K_i \geq d-1$ and height $J+K_i \geq d$ for every i, $1 \leq i \leq d$. By (9) every element of I^2 can be written in the form $\sum_{i=1}^n r_i a_i$ with $r_i \in N$. On the other hand by (8) one necessarily has $r_i \in (K_i, a_i) : \mathfrak{m}$. Therefore

$$r_i \in N \cap [(K_i,a_i):\mathfrak{m}] \subset [N \cap (K_i,a_i)]:\mathfrak{m} = (N \cap K_i, a_i):\mathfrak{m} = J:\mathfrak{m} = I.$$

□

Remark 5.11 (a) If the ideal J of Theorem 5.10 is $\mathfrak{m}$–primary then $N = R$ and the assumption $k \otimes_R J \not\hookrightarrow k \otimes_R N$ is automatically satisfied. In this case the assertion of the theorem is equivalent to [6, 2.2].

(b) In Theorem 5.10, one can also start from the ideal N instead: Indeed let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d with infinite residue field, and let N be an R–ideal with depth $R/N > 0$ that satisfies G_d and AN_{d-2}. Further let $J = (a_1,\ldots,a_d) \subset N$ be an ideal with $a_1 \in \mathfrak{m}N$ and height $J : N = d$. Then J satisfies the assumptions of the theorem. This holds because N satisfies AN_{d-1} ([46, 1.7.a]), hence J satisfies AN_{d-1} ([46, 1.12]) and therefore has sliding depth ([17, 3.4] or [46, 1.8.c]).

(c) Quite generally, if R is a Cohen–Macaulay local ring and N is an R–ideal of grade g satisfying G_s, then N is AN_s provided that either N has sliding depth ([17, 3.3]), or R is Gorenstein and depth $R/N^j \geq \dim R/N - j + 1$ whenever $1 \leq j \leq s-g+1$ ([46, 2.9.a]). The latter assumptions are automatically satisfied if N is strongly Cohen–Macaulay ([14, the proof of 5.3]). Examples of strongly Cohen–Macaulay ideals are ideals in the linkage class of a complete intersection, such as perfect ideals of grade 2 or perfect Gorenstein ideals of grade 3 ([22, 1.11]).

(d) The assumption $k \otimes_R J \not\hookrightarrow k \otimes_R N$ in Theorem 5.10 is essential: Let $R = k[[Y_1,\ldots,Y_d]]$ be a power series ring over an infinite field, and let N be a perfect R–ideal of grade 2 that satisfies G_d and is presented by a $d+1$ by d matrix φ with linear entries in $k[Y_1,\ldots,Y_d]$. We may assume that the entries of the last row of φ generate the maximal ideal $\mathfrak{m}$ of R. Let J be the R–ideal generated by the d maximal minors of φ obtained by deleting one of the first d rows of φ. Then J satisfies all the assumptions of Theorem 5.10, except that $k \otimes_R J \hookrightarrow k \otimes_R N$. Now $I = J : \mathfrak{m} = N$, $\mathcal{R}(I)$ is Cohen–Macaulay and J is a minimal reduction of I, but $\mathrm{r}_J(I) = \mathrm{r}(I) = d-1$ ([48, 3.2 and 3.10], [39, 3.7]).

Corollary 5.12 *Let R be a Cohen–Macaulay local ring with infinite residue field, and let N be an R–ideal satisfying G_s and AN_{s-1}. For a prime ideal $\mathfrak{p}$ of height s with $\mathfrak{p} \notin \mathrm{Ass}(R/N)$, let $J \subset N$ be an R–ideal so that $\nu(J) \leq s \leq$ height $J : N$, $k(\mathfrak{p}) \otimes_R J \not\hookrightarrow k(\mathfrak{p}) \otimes_R N$, and $\nu((N/J)_{\mathfrak{p}}) > 1$ (such an ideal exists if $\nu(N_{\mathfrak{p}}) > 1$, see [46, 1.4]). Further write $I = J : \mathfrak{p}$. Then J is a reduction of I, $\ell(I) = s$, and $\mathrm{r}_J(I) = \mathrm{r}(I) = 1$.*

Proof. First notice that $N_{\mathfrak{p}}$ satisfies AN_{s-1} ([46, 1.10.b]). Thus Remark 5.11(b) shows that after localizing at $\mathfrak{p}$, the assumptions of Theorem 5.10 are satisfied, including condition (c.ii). Now by that theorem, $J_{\mathfrak{p}}$ is a reduction of $I_{\mathfrak{p}}$, $\ell(I_{\mathfrak{p}}) = s$, and $\mathrm{r}_{J_{\mathfrak{p}}}(I_{\mathfrak{p}}) = \mathrm{r}(I_{\mathfrak{p}}) = 1$.

It remains to show that $I^2 = JI$. We only need to verify this equality locally at every associated prime of JI that contains $\mathfrak{p}$. Since $I_{\mathfrak{p}}^2 = J_{\mathfrak{p}} I_{\mathfrak{p}}$ it suffices to prove that JI has no associated prime of height $> s$. Indeed, J satisfies AN_{s-1} ([46, 1.12]) and hence has the sliding depth property ([17, 3.4] or [46, 1.8.c]). In particular depth $R/J \geq \dim R - s$. Thus $I = J : \mathfrak{p}$ has no associated prime of height $> s$, and then by Lemma 5.9, JI has the same property. □

Remark 5.13 If in Corollary 5.12, $\dim R/\mathfrak{p} = 1$, height $N > 0$, and $s \geq 2$, then $\mathcal{R}(I)$ is Cohen–Macaulay. This follows from the corollary, [50, 4.13] (or [46, 4.8]), and [45, 3.6] since depth $R/I \geq 1 = \dim R - s$.

Linkage via complete intersection modules

In the next result we use linkage via complete intersection modules to construct equimultiple modules of reduction number one.

Theorem 5.14 *Let R be a Cohen–Macaulay local ring of dimension ≥ 2 with infinite residue field, and let U be a complete intersection module of rank e that is a vector bundle, but not free. Write $G = U^{**}$ and $E = U :_G \mathfrak{m}$.*

(a) *$E \neq U$, $\mathrm{r}(E) \leq 1$, and $\mathcal{R}(E)$ is Cohen–Macaulay.*

(b) *The following are equivalent:*

(i) *U is a reduction of E.*

(ii) *E is not a complete intersection module.*

(iii) *If R is regular and $U \cong R^{e-1} \oplus M$, then $\nu(\mathfrak{m}M^{**}/M) \geq 2$.*

In either case, U is a minimal reduction of E and $\mathrm{r}_U(E) = \mathrm{r}(E) = 1$.

Proof. Notice that $E \neq U$ and that $\nu(U) = d + e - 1$ since U is not free. Now part (a) follows from (b) together with Theorem 5.4(b) and Corollary 5.6. As to (b), the implications (i) ⇒ (ii) ⇒ (iii) are clear. Thus it suffices to prove that (iii) forces U to be a reduction of E with $\mathrm{r}_U(E) \leq 1$. The remaining assertions follow from Theorem 5.4(b) again.

By Proposition 3.2, there exists a generic Bourbaki ideal $N \cong G''/F$ of G with respect to U in R'' so that N satisfies G_d and either N is a perfect ideal of grade 2 or else $N = R''$. Let $\mathfrak{m}''$ be the maximal ideal of R'', $k = R''/\mathfrak{m}''$, and write J, I for the images of U''/F, E''/F in N. Notice that the natural map $k \otimes_{R''} U'' \to k \otimes_{R''} G''$ is not injective because $\nu(U) > \nu(G)$. Thus the kernel of this map cannot be contained in the image of $k \otimes_{R''} F$, which is a general proper subspace. Therefore $k \otimes_{R''} J \not\hookrightarrow k \otimes_{R''} N$. Also notice that $N/J \cong G''/U''$, hence height $J : N \geq d \geq \nu(J)$. Finally, $I = J :_N \mathfrak{m}'' = J :_{R''} \mathfrak{m}''$, where the first equality follows from the definition of E. The second equation is clear if $d > 2$ or $N = R''$. Otherwise $J \subsetneq N$ are perfect ideals of grade 2 with J a complete intersection ideal, hence $N = J :_{R''} (J :_{R''} N) \supset J :_{R''} \mathfrak{m}''$. Therefore $J :_{R''} \mathfrak{m}'' = J :_N \mathfrak{m}''$ in this case as well.

Now Remark 5.11(b),(c) shows that the assumptions of Theorem 5.10 are satisfied, with the present N playing the role of the ideal N in that theorem, as long as $d \neq 2$. By part (a) of Theorem 5.10, $\mathcal{R}(I)$ is Cohen–Macaulay. Next we use the theorem to prove that J is a minimal reduction of I with $\mathrm{r}_J(I) = 1$.

First assume that $\nu(G/U) > 1$, in which case $\nu(N/J) > 1$. Thus condition (c.ii) of Theorem 5.10 holds if $d \neq 2$, and hence J is a minimal reduction of I with $\mathrm{r}_J(I) = 1$. If on the other hand $d = 2$, then J is a complete intersection of grade 2, hence $I = J :_{R''} \mathfrak{m}''$ cannot be two generated since otherwise $\mathfrak{m}''/J$ would be cyclic, forcing N/J to be cyclic. So the assumptions of Theorem 5.10(b.ii) are satisfied and thus again J is a minimal reduction of I with $\mathrm{r}_J(I) = 1$ (see also [6, 2.2]).

Next assume that $\nu(G/U) = 1$, in which case $U = R^{e-1} \oplus M$ for some R–ideal M. Since U is not free, but is free locally on the punctured spectrum, it follows that M is

Here $\mathrm{br}(E)$ is a non vanishing positive integer called the *Buchsbaum–Rim multiplicity* of φ or of E. (In fact, for this one only needs to assume that coker φ is a module of finite length.)

It would be useful to have practical algorithms to find these multiplicities. We only make the comment that there is a raw estimate of numerical functions. Let $\mathfrak{a}$ be any $\mathfrak{m}$–primary ideal contained in the annihilator of coker φ, for instance $\mathfrak{a} = I_e(\varphi)$. Then

$$\begin{aligned} \mathrm{length}(C(\varphi)_n) &\leq \mathrm{length}(S_n(R^e)/\mathfrak{a}^n S_n(R^e)) \\ &= \binom{e+n-1}{e-1} \cdot \mathrm{length}(R/\mathfrak{a}^n), \end{aligned}$$

since $\mathfrak{a}^n$ annihilates $C(\varphi)_n$. Thus

$$\mathrm{br}(E) \leq \binom{d+e-1}{e-1} \cdot e(\mathfrak{a})$$

where $e(\mathfrak{a})$ is the Hilbert–Samuel multiplicity of the ideal $\mathfrak{a}$.

We are going to make use of a modified value of these numbers. We denote by $\lambda(E)$ the infimum of all $\mathrm{br}(\widetilde{E})$, where x is an element of $\mathfrak{m}$ not contained in any of the minimal primes of R, $\overline{R} = R/(x)$, and $\widetilde{E}$ is the image of E in $\overline{R}^e$. We first want to compare this new invariant to $\mathrm{br}(E)$.

Proposition 5.15 *If R is a Cohen-Macaulay local ring with infinite residue field then* $\lambda(E) \leq \mathrm{br}(E)$.

The proof of the proposition requires a lemma.

Lemma 5.16 *Let $(A, \mathfrak{n})$ be a one-dimensional Cohen-Macaulay local ring, M a finitely generated R–module, and $F \subset \mathfrak{n}M$ a free submodule. Then* $\mathrm{length}(M/F) \geq e(M)$.

Proof. We may assume that $e = \mathrm{rank}\, F > 0$ and that $\mathrm{length}(M/F) < \infty$. Now M has rank $e > 0$ as well. Factoring out the torsion of M we reduce to the case where M is torsionfree. Furthermore we may suppose that A has an infinite residue field. Let (x) be a minimal reduction of $\mathfrak{n}$ and U a minimal reduction of M. By Proposition 2.3, U is necessarily free of rank e. Now the free module xU is a reduction of $\mathfrak{n}M$. Therefore

$$\mathrm{length}(\mathfrak{n}M/F) \geq \mathrm{length}(\mathfrak{n}M/xU),$$

as can be easily seen like in the case of ideals. Thus $\mathrm{length}(M/F) \geq \mathrm{length}(M/xU) \geq \mathrm{length}(M/xM) \geq e(M)$. □

Proof of Proposition 5.15. First notice that $d \geq 2$ since E is not free. Let $\mathfrak{m}$ denote the maximal ideal of R. Factoring out a free direct summand of R^e contained in E we may assume that $E \subset \mathfrak{m}R^e$. Write $E = \sum_{i=1}^n Ra_i$, let $z_{ij},\ 1 \leq j \leq d+e-1,\ 1 \leq i \leq n$, be variables and $R'' = R(\{z_{ij}\}),\ E'' = R'' \otimes_R E,\ x_j = \sum_{i=1}^n z_{ij} a_i$. Consider the R''–submodules $U = \sum_{j=1}^{d+e-2} R'' x_j \subset V = \sum_{j=1}^{d+e-1} R'' x_j$ of E'', and write $F = V/U \subset M = R''^e/U$. Further let $A = R''/\mathrm{ann}_{R''}(M)$ and let $\mathfrak{n}$ be the maximal ideal of

A. Notice that $F \subset \mathfrak{n}M$ and $\dim A \geq 1$. There exists an R–regular element x that is part of a minimal generating set of $\mathfrak{n}$. Write $\overline{R} = R/(x)$, $\overline{R''} = R''/(x)$, and let $\widetilde{E}$ be the image of E in $\overline{R}^e$, and $\widetilde{U}, \widetilde{V}, \widetilde{E''}$ the images of U, V, E'' in $\overline{R''}^e$. Obviously $M/F = R''^e/V$ and $M/xM = \overline{R}''^e/\widetilde{U}$.

According to Proposition 2.3, $\ell(E) \leq d+e-1$ and $\ell(\widetilde{E}) \leq d+e-2$. Hence by the genericity of V and U, V is a reduction of E'' and $\widetilde{U}$ is a reduction of $\widetilde{E''}$. In particular $M/F = R''^e/V$ and $M/xM = \overline{R}''^e/\widetilde{U}$ have finite length. By the latter, $\widetilde{U}$ is a complete intersection module. Furthermore, $\dim M \leq \dim M/xM + 1 = 1$, so U is a complete intersection module as well. It follows that M is Cohen-Macaulay of dimension 1. Thus A has the same property. In particular Ax is a minimal reduction of $\mathfrak{n}$. Since M is a faithful A-module and $F \subset M$ is a cyclic submodule of finite colength, it follows that F is free over A. As $F \subset \mathfrak{n}M$, Lemma 5.16 now implies

$$\operatorname{length}(M/F) \geq e(M).$$

But $e(M) = \operatorname{length}\,(M/xM)$ by the Cohen-Macaulayness of M and our choice of x. Therefore

$$\operatorname{length}(R''^e/V) \geq \operatorname{length}(\overline{R''}^e/\widetilde{U}).$$

On the other hand, since V is a reduction of E'', $\operatorname{br}(E) = \operatorname{br}(E'') = \operatorname{br}(V)$ ([33, 4.5.iv] or [34, 5.3.i]), and since V is a complete intersection module, $\operatorname{br}(V) = \operatorname{length}(R''^e/V)$ ([5, 4.5]). Likewise, $\operatorname{br}(\widetilde{E}) = \operatorname{br}(\widetilde{E''}) = \operatorname{br}(\widetilde{U}) = \operatorname{length}(\overline{R''}^e/\widetilde{U})$.

Thus $\operatorname{br}(E) \geq \operatorname{br}(\widetilde{E})$, showing that $\operatorname{br}(E) \geq \lambda(E)$. □

We mention in passing that Lemma 5.16 (together with [2, Corollary 1]) also shows the following: Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, let $1 \leq e \leq m$, and let ϕ an e by m matrix with entries in $\mathfrak{m}$ so that grade $I_e(\phi) = m - e + 1$. Then for every submatrix φ of ϕ, $e(R/I_e(\varphi)) \leq e(R/I_e(\phi))$.

Theorem 5.17 *Let R be a Noetherian local ring and E an ideal module that is a vector bundle, but not free. Then the special fiber $\mathcal{F}(E)$ has multiplicity at most $\lambda(E)$.*

Proof. Write $d = \dim R$ and $e = \operatorname{rank} E$. Again $d \geq 2$ since E is not free. For an element x contained in the maximal ideal but not in any minimal prime of R write $\overline{R} = R/(x)$ and $\overline{\varphi} = \mathrm{id}_{\overline{R}} \otimes \varphi$. Reduce the exact sequence (10) modulo x to obtain the exact sequence

$${}_xC(\varphi) \longrightarrow \mathcal{R}(E)/x\mathcal{R}(E) \longrightarrow \mathcal{S}(\overline{R}^e) \longrightarrow C(\overline{\varphi}) \to 0.$$

We decompose this sequence into two short exact sequences

$${}_xC(\varphi) \longrightarrow \mathcal{R}(E)/x\mathcal{R}(E) \longrightarrow \mathcal{R}(\widetilde{E}) \to 0, \tag{11}$$

and

$$0 \to \mathcal{R}(\widetilde{E}) \longrightarrow \mathcal{S}(\overline{R}^e) \longrightarrow C(\overline{\varphi}) \to 0,$$

where we denote by $\widetilde{E}$ the image of $\overline{\varphi}$. Note that with the hypotheses, coker $\overline{\varphi}$ is a nontrivial $\overline{R}$–module of finite length. As a consequence, $C(\overline{\varphi})$ is a graded module, with a Hilbert polynomial of degree $d+e-2$ and multiplicity $\mathrm{br}(\widetilde{E})$.

Observe that multiplication by x on $C(\varphi)$ induces the exact sequence of graded modules

$$0 \to {}_xC(\varphi) \longrightarrow C(\varphi) \xrightarrow{x} C(\varphi) \longrightarrow C(\overline{\varphi}) \to 0.$$

We then have that ${}_xC(\varphi)$ has precisely the same Hilbert function as $C(\overline{\varphi})$. Finally, tensoring (11) by the residue field k of R, we obtain the sequence of special fibers

$$k \otimes {}_xC(\varphi) \longrightarrow \mathcal{F}(E) \longrightarrow \mathcal{F}(\widetilde{E}) \to 0.$$

As $\ell(E) = d+e-1$ by Proposition 5.2 and $\ell(\widetilde{E}) \leq d+e-2$ by Proposition 2.3, the Hilbert polynomial of $\mathcal{F}(E)$ has degree $d+e-2$, whereas the Hilbert polynomial of $\mathcal{F}(\widetilde{E})$ has degree at most $d+e-3$. This implies that the leading coefficient of the Hilbert polynomial of $\mathcal{F}(E)$ is bounded by the multiplicity of ${}_xC(\varphi)$, as desired. □

Corollary 5.18 *Let R and E be as in* Theorem 5.17 *and suppose further that the residue field of R has characteristic zero. If $\mathcal{F}(E)$ satisfies the condition S_1 then* $\mathrm{r}(E) \leq \lambda(E) - 1$. *If in addition R is Cohen-Macaulay with infinite residue field then $r(E) \leq \mathrm{br}(E) - 1$.*

Proof. With the above bound on the multiplicity of $\mathcal{F}(E)$ this follows directly from [52]. □

The hypothesis on the condition S_1 always holds if R is a positively graded domain over a field and E is graded, generated by elements of the same degree. In this case $\mathcal{F}(E)$ embeds into $\mathcal{R}(E)$.

Example 5.19 Let $R = k[x_1, \ldots, x_d]$ and $E = (x_1, \ldots, x_d)^{\oplus e}$, $d \geq 2, e > 0$. Applying the argument above we find that $\mathcal{F}(E)$ has multiplicity bounded by $\binom{d+e-2}{e-1}$. This is much too large as a bound for $\mathrm{r}(E)$. In fact, $\mathcal{F}(E) \cong k[x_iy_j \mid 1 \leq i \leq d,\ 1 \leq j \leq e]$. This last ring is studied in [8] and has for reduction number $\min\{d-1, e-1\}$ (in any characteristic).

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