Congruences for $r_s(n)$ Modulo $2s$

Samuel S. Wagstaff, Jr.

Department of Computer Science and Center for Education and Research in Information Assurance and Security, 1315 Recitation Building, Purdue University, West Lafayette, IN 47907-1315, USA

Abstract

We determine $r_s(n)$ modulo $2s$ when s is a prime or a power of 2. For general s, we prove a congruence for $r_s(n)$ modulo the largest power of 2 dividing $2s$.

Key words: Sums of squares, congruence.

Let $r_s(n)$ denote the number of ways to write an integer n as the sum of s squares of integers, that is, $r_s(n)$ is the number of solutions to

$$n = x_1^2 + x_2^2 + \cdots + x_s^2 \tag{1}$$

in integers x_i. Clearly, $r_s(0) = 1$.

Exact formulas for $r_s(n)$ are known for various small s. These include

$$r_2(n) = 4 \sum_{2\ell+1|n} (-1)^\ell, \tag{2}$$

$$r_4(n) = 8 \cdot 3^\delta \sum_{2\ell+1|n} (2\ell+1), \text{ where } \delta = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

$$r_8(n) = 16 \sum_{d|n} (-1)^{n+d} d^3. \tag{4}$$

The formulas (2), (3) and (4) are derived by equating the coefficients in well known identities of Jacobi. See, for example, page 307 of Smith [3], or Chapter IX of Hardy [2], or page 121 of Grosswald [1].

Email address: `ssw@cerias.purdue.edu` (Samuel S. Wagstaff, Jr.).

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Similar formulas are known for even s up to about 24. Formulas for odd $s > 1$ are more complicated. They may involve class numbers and, when $s > 8$, coefficients of cusp forms.

In this note we prove congruences for $r_s(n)$ modulo $2s$ for infinitely many s. It is clear from (2), (3) and (4) that for all $n \geq 1$ we have

$$4|r_2(n), \qquad 8|r_4(n), \qquad 16|r_8(n).$$

In other words, $r_s(n) \equiv 0 \pmod{2s}$ for $s = 2, 4, 8$, and for all $n \geq 1$. This congruence also holds for $s = 1$.

However, it is not true that $r_s(n) \equiv 0 \pmod{2s}$ for all s and $n \geq 1$. For example, $r_3(27) = 32 \equiv 2 \pmod 6$, $r_5(20) = 752 \equiv 2 \pmod{10}$, $r_6(3) = 160 \equiv 4 \pmod{12}$ and $r_9(6) = 7932 \equiv 12 \pmod{18}$. The following theorems explain these values.

Theorem 1 *Let p be a prime and k and n be positive integers. Let $s = p^k$. If $p = 2$, then $r_s(n) \equiv 0 \pmod{2s}$. If p is odd, then*

$$r_s(n) \equiv \begin{cases} 2 \pmod{2p} & \textit{if } n = st^2 \textit{ for some positive integer } t, \\ 0 \pmod{2p} & \textit{otherwise.} \end{cases}$$

Proof. Suppose first that $p = 2$ and $s = 2^k$. We prove by induction on k that $r_s(n) \equiv 0 \pmod{2s}$. Formulas (2), (3) and (4) give the result for $k = 1$, 2 and 3.

If we let

$$\vartheta(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}$$

denote the the generating function of the squares, then it is well known that

$$(\vartheta(q))^s = 1 + \sum_{n=1}^{\infty} r_s(n)q^n = \sum_{n=0}^{\infty} r_s(n)q^n$$

is the generating function for $r_s(n)$. Now $(\vartheta(q))^{2s} = ((\vartheta(q))^s)^2$, so for $n \geq 0$,

$$\sum_{n=0}^{\infty} r_{2s}(n)q^n = \left(\sum_{i=0}^{\infty} r_s(i)q^i\right)^2.$$

When we equate the coefficients of q^n on each side we find, for $n \geq 0$,

$$r_{2s}(n) = \sum_{i=0}^{n} r_s(i) r_s(n-i). \tag{5}$$

Assume by induction that $r_s(n) \equiv 0 \pmod{2s}$ for a given s and all $n > 0$. Then Equation (5) implies that $r_{2s}(n) \equiv 2r_s(0)r_s(n) \pmod{4s^2}$. Using $s \geq 2$, $r_s(0) = 1$ and the inductive hypothesis again, we find $r_{2s}(n) \equiv 0 \pmod{4s}$, and the proof is complete for $p = 2$.

Now suppose that p is an odd prime and $s = p^k$ with $k \geq 1$. If the s-tuple $(x_1, \ldots, x_s)$ is a solution to (1) counted in $r_s(n)$, then at least one $x_i \neq 0$. Let x_j be the first nonzero one. Then the pairing

$$(x_1, \ldots, x_j, \ldots, x_s) \longleftrightarrow (x_1, \ldots, -x_j, \ldots, x_s)$$

pairs distinct solutions to (1) and shows that their number is even, that is, $r_s(n) \equiv 0 \pmod 2$. (In fact, this pairing shows that $r_s(n)$ is even for any positive integers s and n.)

Let σ denote the permutation of the s-tuple $(x_1, \ldots, x_s)$ that rotates the components one position to the left. Let G be the permutation group generated by σ. Clearly, G is cyclic of order s. When $(x_1, \ldots, x_s)$ is a solution to (1) so is every permutation of this s-tuple. The $r_s(n)$ solutions to (1) are partitioned by the action of G into disjoint orbits. The size of the orbit of $(x_1, \ldots, x_s)$ under the action of G divides the order of G, and hence is a power of p. The size is 1 if and only if $x_i = t$ for $i = 1, \ldots, s$ and some t. If $n = st^2$, then the orbits of the two s-tuples $(t, t, \ldots, t)$ and $(-t, -t, \ldots, -t)$ each have size 1. In all other cases the size of the orbit is a multiple of p. Therefore, the number of solutions to (1) is a multiple of p when n does not have the form st^2, and it is 2 more than a multiple of p when $n = st^2$ for some positive integer t. This completes the proof.

Corollary 2 *If s is an odd prime and n is a positive integer, then*

$$r_s(n) \equiv \begin{cases} 2 \pmod{2s} & \text{if } n = st^2 \text{ for some positive integer } t, \\ 0 \pmod{2s} & \text{otherwise.} \end{cases}$$

Theorem 3 *If $s = 2^k m > 0$, with m odd and $k \geq 0$, then for all $n > 0$ we have*

$$r_s(n) \equiv 0 \pmod{2^{k+1}}.$$

Proof. If $m = 1$, this theorem is just the first part of Theorem 1. Therefore, we may assume $m \geq 3$.

Use induction on k. As noted in the proof of Theorem 1, $r_s(n)$ is even for any positive integers s and n. This shows the base step $k = 0$. Assume the congruence holds for some k and some m, that is, for some s. We prove it for $k + 1$ and the same m, that is, for $2s$. The convolution (5) applies and shows that $r_{2s}(n) \equiv 2r_s(n) \pmod{2^{2(k+1)}}$. Since $2(k + 1) \geq k + 2$ and 2^{k+1} divides $r_s(n)$, we have $r_{2s}(n) \equiv 0 \pmod{2^{k+2}}$, and the proof is complete.

Remark. Tables of $r_s(n)$ suggest that Theorems 1 and 3 describe *all* congruences modulo a divisor of $2s$ satisfied by $r_s(n)$ for all $n > 0$. For example, when $s = 9$, $r_9(n) \equiv 0, 2, 6, 8, 12, 14 \pmod{18}$ for $n = 1, 225, 3, 9, 6, 81$, respectively. Likewise, $r_{15}(n) \equiv 0, 2, 4, \ldots, 28 \pmod{30}$ when $n = 1, 540, 120, 5, 60, 3, 10, 30, 330, 70, 9, 135, 25, 90, 15$, respectively. Also, $r_{18}(n) \equiv 0, 4, 8, \ldots, 32 \pmod{36}$ for $n = 1, 18, 180, 3, 9, 45, 6, 36, 90$, respectively. In each of these examples, the value of n is the smallest one for which $r_s(n)$ lies in the specified congruence class.

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