Ramanujan's Taxicab Number and its Ilk

Samuel S. Wagstaff, Jr *
Center for Education and Research in Information Assurance and Security
and Department of Computer Sciences
Purdue University
West Lafayette, IN 47907-1398 USA
Email: ssw@cerias.purdue.edu

Abstract

We discuss properties of the Hardy-Ramanujan taxicab number, 1729, and similar numbers. The similar numbers include Carmichael numbers, Lucas Carmichael numbers, sums of cubes and integers of the form $b^n \pm 1$.

2020 *Mathematics Subject Classification*. Primary 11A25; Secondary 11D25, 11Y55.
Key words and phrases. Ramanujan taxicab number, Sum of two cubes, Carmichael number, Lucas Carmichael number.

A *Carmichael number* is an odd square free composite integer c such that $p - 1 \mid c - 1$ for every prime factor p of c. (See [5] or [8, Theorem 3.36].) The Hardy-Ramanujan taxicab number 1729 is a Carmichael number with several interesting properties. We wondered which other Carmichael numbers share these properties. See [2] for more properties shared by Carmichael numbers with other numbers.

As Ramanujan noted, 1729 is the smallest integer that is the sum of two cubes of positive integers in two different ways. Many Carmichael numbers are the sum of two positive cubes in at least one way. The first few are: 1729, 15841, 46657, 126217, 188461, 1082809. However, 1729 is the only Carmichael number $< 10^{21}$ which is the sum of two cubes of positive integers in two or more different ways. We used the methods of Bernstein [4] to generate the sums of two cubes $< 10^{21}$ and compared them to Pinch's list [7] of the 20138200 Carmichael numbers $< 10^{21}$. (The smaller Carmichael number 1105 can be written as the sum of two *squares* in more ways than any smaller number. Of course, $1729 = 7 \cdot 13 \cdot 19$ is not the sum of two squares.) We don't know whether any larger Carmichael number is the sum of two cubes of positive integers in two or more different ways.

The Cunningham Project [3] factors numbers of the form $b^n \pm 1$ for small bases b. The number $1729 = 12^3 + 1$ has this form. We tested the Carmichaels $< 10^{21}$ for having the shape $b^n \pm 1$ with $n > 1$, $b > 1$ and found only the examples in Table 1. Note that all b are multiples of 6.

It is curious that all have the form $b^n + 1$, and none have the form $b^n - 1$. Here is a possible explanation. Suppose a Carmichael number c has the form $b^n - 1$ with integers $b > 1$, $n > 1$. Then b is even because c is odd. Also $4 \mid b^n$ because $n > 1$, so $c \equiv -1 \pmod 4$.

*This work was supported by the CERIAS Center at Purdue University.

Table 1: Carmichael numbers $< 10^{21}$ with the form $b^n \pm 1$, $b > 1$, $n > 1$.

Carmichael number	Form $b^n \pm 1$
1729	$12^3 + 1$
46657	$6^6 + 1$
2433601	$1560^2 + 1$
2628073	$138^3 + 1$
19683001	$270^3 + 1$
67371265	$8208^2 + 1$
110592000001	$4800^3 + 1$
351596817937	$592956^2 + 1$
422240040001	$649800^2 + 1$
432081216001	$7560^3 + 1$
2116874304001	$12840^3 + 1$
3176523000001	$14700^3 + 1$
18677955240001	$4321800^2 + 1$
458631349862401	$21415680^2 + 1$
286245437364810001	$535019100^2 + 1$
312328165704192001	$678480^3 + 1$

If a prime $p = 4k + 1$ divides c, then $4 \mid p - 1$ and $p - 1 \mid c - 1$, so $c \equiv 1 \pmod 4$. This contradiction shows that every prime factor of c must be $\equiv -1 \pmod 4$. Furthermore, c must have an odd number of such prime factors since $c \equiv -1 \pmod 4$. Some Carmichael numbers, like $8911 = 7 \cdot 19 \cdot 67$, and some $b^n - 1$, like $2^{10} - 1 = 3 \cdot 11 \cdot 31$, have this form, but both types are rare. Hence the smallest example of a Carmichael number $c = b^n - 1$ is probably quite large.

The argument just given shows that no Carmichael number has the form $b^4 - 1$, since its factor $b^2 + 1$ is the sum of two squares, so cannot be odd, square free and have a prime factor $\equiv -1 \pmod 4$. I thank Carl Pomerance for noticing this fact.

In a similar vein, we can study Lucas Carmichael numbers. A *Lucas Carmichael number* is an odd square free composite integer m such that $p+1 \mid m+1$ for every prime factor p of m. Donovan Johnson (see the link in OEIS [6] Sequence A006972) computed the first 10000 Lucas Carmichael numbers, up to 1008003203999. For comparison, the 10000th Carmichael number is 1713045574801. Using his list, we tested each number for having the form $b^n \pm 1$. We found that 164 of the first 10000 Lucas Carmichael numbers have this form. For every one of these numbers the form was $b^2 - 1$. None had the form $b^n + 1$. None had the form $b^n - 1$ with $n > 2$. The first eleven examples are shown in Table 2, as well as the 162nd through 164th ones. Most, but not all, b are multiples of 6.

An argument similar to that given above for Carmichael numbers, with $+1$ and -1 swapped, explains why Lucas Carmichael numbers with the form b^n+1 might be rare. Some Lucas Carmichael numbers, like $80189 = 17 \cdot 53 \cdot 89$, and some b^n+1, like $2^{22}+1 = 5 \cdot 397 \cdot 2113$, have all prime factors $\equiv 1 \pmod 4$, but such numbers are rare. It is a mystery why so many Lucas Carmichael numbers are 1 less than a square, while none are $b^n - 1$ with $n > 2$.

Table 2: Some Lucas Carmichael numbers with the form $b^n \pm 1$, $b > 1$, $n > 1$.

Lucas Carmichael number	Form $b^n \pm 1$
399	$20^2 - 1$
2915	$54^2 - 1$
7055	$84^2 - 1$
63503	$252^2 - 1$
147455	$384^2 - 1$
1587599	$1260^2 - 1$
1710863	$1308^2 - 1$
2249999	$1500^2 - 1$
2924099	$1710^2 - 1$
6656399	$2580^2 - 1$
9486399	$3080^2 - 1$
$\vdots$	$\vdots$
919354968899	$958830^2 - 1$
933967616399	$966420^2 - 1$
938844723599	$968940^2 - 1$

References

[1] W. R. Alford, Andrew Granville, and Carl Pomerance: There are infinitely many Carmichael numbers, Ann. Math., 139, 703–722 (1994). doi:10.2307/2118576, JSTOR 2118576

[2] William Banks, Carrie Finch, Florian Lucas, Carl Pomerance, and Pantelimon Stănică: Sierpiński and Carmichael numbers, Trans. Amer. Math. Soc. 367, 355–376 (2015)

[3] John Brillhart, D. H. Lehmer, J. L. Selfridge, Bryant Tuckerman, and S. S. Wagstaff, Jr: Factorizations of $b^n \pm 1$, $b = 2, 3, 5, 6, 7, 10, 11, 12$ up to high powers, Second edition, Amer. Math. Soc., Providence, Rhode Island (1988)

[4] Daniel J. Bernstein: Enumerating solutions to $p(a) + q(b) = r(c) + s(d)$, Math. Comp., 70, 389–394 (2001)

[5] A. Korselt: Probléme chinois, L'Intermédiaire des Mathématiciens 3, 142–143 (1899)

[6] OEIS Foundation Inc.: The On-Line Encyclopedia of Integer Sequences, (2023). Available at `https://oeis.org`

[7] Richard Pinch: The Carmichael numbers up to 10^{21}, Proceedings of the Conference on Algorithmic Number Theory, 46, Turku, Finland: Turku Center for Computer Science, 129–131, (2007)

[8] S. S. Wagstaff, Jr: The Joy of Factoring, volume 68 of Student Mathematical Library. Amer. Math. Soc., Providence, Rhode Island, (2013)