SINGULARITIES AND HOLONOMICITY OF BINOMIAL D-MODULES

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ABSTRACT. We study binomial D-modules, which generalize A-hypergeometric systems. We determine explicitly their singular loci and provide three characterizations of their holonomicity. The first of these is an equivalence of holonomicity and L-holonomicity for these systems. The second refines the first by giving more detailed information about the L-characteristic variety of a non-holonomic binomial D-module. The final characterization states that a binomial D-module is holonomic if and only if its corresponding singular locus is proper.

1. INTRODUCTION

Binomial ideals in a polynomial ring over a field enjoy many special properties that set them apart from more general ideals. For example, work of Eisenbud and Sturmfels [ES96] shows that toric ideals can be viewed as basic building blocks of binomial ideals. Extending this point of view to D-modules, binomial D-modules (Definition 2.3) were introduced in [DMM10b] as a generalized framework to study systems of hypergeometric differential equations; here, the pendant to the toric ideals are the A-hypergeometric differential equations of Gelfand, Graev, Kapranov and Zelevinsky [GGZ87, GKZ89, GKZ90].

As in the polynomial case, binomial D-modules have some unusual properties. For instance, a binomial D-module is holonomic if and only if it has a finite dimensional solution space; while the forward implication in the previous statement is true in general, the converse certainly is not.

The goal of this article is to provide more results in this vein, further showing how special binomial D-modules are within the class of all D-modules. For this purpose, we study the characteristic variety and singular locus of a binomial D-module, and use our conclusions to obtain new characterizations of holonomicity for these objects.

Our main result is that a binomial D-module on $\mathbb{C}^n$ is holonomic if and only if its restriction to $(\mathbb{C}^*)^n$ is holonomic (Theorem 3.1), if and only if its singular locus is a proper subvariety of $\mathbb{C}^n$ (Theorem 4.2). As before, the forward implications are always true, but the converses fail in general, even in the simplest instances (Examples 3.3 and 4.3).

A strong motivation for the statements in this paper comes from our companion article [BMW13]. In that work, the results here are used to obtain conclusions about classical systems of hypergeometric differential equations; see Remark 3.2 for more details.

Outline. In Section 2, we introduce concepts and notation about D-modules that will be used throughout. In Section 3, we prove Theorem 3.1 using ideas from [SW08]. In Section 4, we prove

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Theorem 4.2 by referring to a result from [BMW13] and we give a combinatorial description of the singular locus of a binomial D-module along the lines of previous work by Gelfand, Kapranov and Zelevinsky, and Adolphson.

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2. PRELIMINARIES

2.1. **Set-up.** We let $d \leq n$ stand for two elements of the set of natural numbers $\mathbb{N} = 0, 1, 2, \ldots$

Convention 2.1. Throughout this article, $A = [\, a_1\, a_2\, \cdots\, a_n\,]$ is an integer $d \times n$ matrix such that $\mathbb{Z} \cdot A = \mathbb{Z}^d$ as lattices, and that there exists $h \in \mathbb{Q}^d$ such that $h \cdot a_i > 0$ for $i = 1, \ldots, n$.

Let X be affine n-space over $\mathbb{C}$, with coordinates $x_1, \ldots, x_n$. The Weyl algebra D is the ring of differential operators on X; it is generated by the multiplication operators $x_1, \ldots, x_n$ and the differentiation operators $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n}$, subject to the Leibniz rule $\partial_j x_i - x_i \partial_j = \delta_{ij}$ (the Kronecker delta).

2.2. **Holonomicities and singular locus.** A *projective weight vector on* D is $L = (L_x, L_\partial) \in \mathbb{Q}^n \times \mathbb{Q}^n$ such that $L_x + L_\partial = c \cdot \mathbf{1}_n := c \cdot (1, \ldots, 1)$ for some constant $c > 0$. This determines an increasing filtration L on D by $L^k D := \mathbb{C} \cdot \{x^u \partial^v \mid L \cdot (u, v) \leq k\}$ for $k \in \mathbb{Q}$. Set $L^{<k} D := \bigcup_{\ell < k} L^\ell D$. Since $c > 0$, the associated graded ring $\operatorname{gr}^L D$ is isomorphic to the coordinate ring of $T^*X \cong \mathbb{C}^{2n}$, which is a polynomial ring in $2n$ variables. For any P in $L^k D \smallsetminus L^{<k} D$, set $\operatorname{in}_L(P) := P + L^{<k} D \in \operatorname{gr}^{L,k} D := L^k D / L^{<k} D \subseteq \operatorname{gr}^L D$ and $\deg^L(P) := k$. By a slight abuse of notation, set $x_i := \operatorname{in}_L(x_i)$ and $\xi_i := \operatorname{in}_L(\partial_i)$, where (x, ξ) are coordinates on T^*X.

For a left D-ideal I, set $\operatorname{gr}^L(I) := \langle \operatorname{in}_L(P) \mid P \in I \rangle \subseteq \operatorname{gr}^L(D)$. The L-*characteristic variety* of the module D/I is

$$\operatorname{Char}^L(D/I) := \operatorname{Var}(\operatorname{gr}^L(I)) \subseteq T^*X \cong \mathbb{C}^{2n}. \tag{2.1}$$

The projective weight vector $F = (\mathbf{0}_n, \mathbf{1}_n) := (0, \ldots, 0, 1, \ldots, 1) \in \mathbb{Q}^{2n}$ induces the *order filtration* on D. The F-characteristic variety of a D-module is usually called its characteristic variety. The *singular locus* of D/I, denoted $\operatorname{Sing}(D/I)$, is the projection of $\operatorname{Char}^F(D/I) \smallsetminus \operatorname{Var}(\xi_1, \ldots, \xi_n)$ onto X, and as such, it is a closed subvariety of X.

The *divisorial singular locus* of D/I, denoted by $\operatorname{Sing}^1(D/I)$, is the codimension at most one part of $\operatorname{Sing}(D/I)$. From the point of view of (classical) holomorphic solutions of systems of differential equations, there is no difference between $\operatorname{Sing}(D/I)$ and $\operatorname{Sing}^1(D/I)$ because the codimension two singularities of holomorphic functions can be removed.

For a left D-ideal I, $\dim(\operatorname{Char}^F(D/I)) \geq n$ by Bernstein's inequality [Ber72] (see also [Smi01]); D/I is *holonomic* if equality holds.

Definition 2.2. The D-module D/I is L-*holonomic* if $\operatorname{Char}^L(D/I)$ is empty or has dimension n. The *rank* of D/I is $\operatorname{rank}(D/I) := \dim_{\mathbb{C}(x)} \mathbb{C}(x) \otimes_{\mathbb{C}[x]} D/I$.

2.3. **Binomial D-modules.** We recall here binomial D-modules and the structure of their L-characteristic varieties from [DMM10b].

The matrix A determines a $(\mathbb{C}^*)^d$-action on X by

$$t \diamond p = (t^{a_1}p_1, \dots, t^{a_n}p_n) \text{ for } t = (t_1, \dots, t_d) \in (\mathbb{C}^*)^d \text{ and } p = (p_1, \dots, p_n) \in X.$$

This action passes to the Weyl algebra D via

$$t \diamond x_i := t^{a_i}x_i \quad \text{and} \quad t \diamond \partial_i = t^{-a_i}\partial_i \quad \text{for } i = 1, \dots, n.$$

Let $A = [a_{ij}]$ be as in Convention 2.1. The *Euler operators for A* are

$$E_i := \sum_{j=1}^{n} a_{ij}x_j\partial_j \quad \text{for } i = 1, \dots, d. \tag{2.2}$$

We write E_A for $E_1, \dots, E_d$, and for $\beta \in \mathbb{C}^d$, we denote by $E_A - \beta$ the sequence $E_1 - \beta_1, \dots, E_d - \beta_d$. Let $I \subseteq \mathbb{C}[\partial_1, \dots, \partial_n] = \mathbb{C}[\partial]$ be a binomial ideal, that is, an ideal generated by binomials and monomials. We assume that I is equivariant with respect to the $(\mathbb{C}^*)^d$-action on $\mathbb{C}[\partial]$ induced by A.

Definition 2.3. Given $\beta \in \mathbb{C}^d$, a *binomial D-module* is of the form

$$\frac{D}{(I, E_A - \beta)} := \frac{D}{D \cdot I + D \cdot (E_A - \beta)}.$$

The (very special) binomial ideal

$$I_A := \langle \partial^u - \partial^v \mid u, v \in \mathbb{N}^n,\ Au = Av \rangle \subseteq \mathbb{C}[\partial] \tag{2.3}$$

is called the *toric ideal* associated to A. The left D-ideal

$$H_A(\beta) := D \cdot (I_A, E_A - \beta)$$

is called an *A-hypergeometric system*, and $D/H_A(\beta)$ is called an *A-hypergeometric D-module*.

2.4. **Toral and Andean components.** Every associated prime of a binomial ideal is also binomial, and every prime binomial ideal is isomorphic to a toric ideal up to a rescaling of the variables [ES96]. We review from [DMM10b] how A-hypergeometric systems play a similarly fundamental role in the theory of binomial D-modules.

The action of the torus $(\mathbb{C}^*)^d$ on $\mathbb{C}[\partial]$ defines the *A-grading*, with $\deg(x_i) = -\deg(\partial_i) := a_i$. A binomial ideal $I \subseteq \mathbb{C}[\partial]$ is torus equivariant if and only if it is A-graded. If $M = \bigoplus_{\alpha \in \mathbb{Z}^d} M_\alpha$ is an A-graded $\mathbb{C}[\partial]$-module, then the set of *quasidegrees* of M is

$$\operatorname{qdeg}(M) := \overline{\{\alpha \in \mathbb{Z}^d \mid M_\alpha \neq 0\}}^{\text{Zariski}} \subseteq \mathbb{C}^d,$$

where the closure is taken in the Zariski topology of $\mathbb{C}^d$.

Let $\mathscr{C}$ be a primary component of an A-graded binomial ideal $I \subseteq \mathbb{C}[\partial]$, which can be chosen to be binomial by [ES96]. If the A-graded Hilbert function of $\mathbb{C}[\partial]/\sqrt{\mathscr{C}}$ is bounded, then the component $\mathscr{C}$, along with its corresponding associated prime $\sqrt{\mathscr{C}}$, is *toral*; otherwise, they are *Andean*. Examples and more details can be found in [DMM10a].

Theorem 2.4. [DMM10b, Theorem 6.3] *Let $I \subseteq \mathbb{C}[\partial]$ be an A-graded binomial ideal. The binomial D-module $D/(I, E_A - \beta)$ is holonomic if and only if $-\beta$ lies outside the union of the sets* $\operatorname{qdeg}(\mathbb{C}[\partial]/\mathscr{C})$*, running over the Andean components $\mathscr{C}$ of the binomial ideal I.* □

2.5. L-**Characteristic varieties.** We next reproduce results of Schulze and Walther [SW08] that set-theoretically describe the L-characteristic variety of hypergeometric ideals.

Let A and $h = (h_1, \ldots, h_d)$ be as in Convention 2.1, and let $L = (L_x, L_\partial) \in \mathbb{Q}^{2n}$ be a projective weight vector on D. Choose $\varepsilon > 0$ such that $h \cdot a_i + \varepsilon L_{\partial_i} > 0$ for $i = 1, \ldots, n$, and denote by H_ε the hyperplane in $\mathbb{P}^d_{\mathbb{Q}}$ given by $\{(y_0 : y_1 : \cdots : y_d) \in \mathbb{P}^d_{\mathbb{Q}} \mid \varepsilon\, y_0 + h_1 y_1 + \cdots + h_d y_d = 0\}$. The *$L$-polyhedron of A* is the convex hull of $\{(1 : \mathbf{0}_d), (L_{\partial_1} : a_1), \ldots, (L_{\partial_n} : a_n)\}$ in the affine space $\mathbb{P}^d_{\mathbb{Q}} \smallsetminus H_\varepsilon$. The *$L$-umbrella of A*, denoted $\Phi(A, L)$, is the set of faces of the L-polyhedron of A that do not contain $(1 : \mathbf{0}_d)$.

Let $\tau \in \Phi(A, L)$ and identify τ with the subset of $\{1, \ldots, n\}$ indexing the columns of A belonging to τ. Whenever it is convenient, view τ as the set $\{a_i \mid i \in \tau\}$, or as the matrix whose columns are a_i for $i \in \tau$. Denote by $\bar{\tau}$ the set $\{1, \ldots, n\} \smallsetminus \tau$.

Let C_τ denote the conormal space to the orbit under the torus action of the point $\mathbf{1}_\tau$ in $\mathbb{C}^n$ whose coordinates indexed by τ are equal to 1 and those indexed by $\bar{\tau}$ are equal to 0. Writing $x\xi := (x_1\xi_1, \ldots, x_n\xi_n)$ and $\xi_\tau := \prod_{j \in \tau} \xi_j$, the Zariski closure of C_τ, denoted $\overline{C_\tau}$, is equal to the Zariski closure in $T^*(\mathbb{C}^n)$ of the variety in $T^*(\mathbb{C}^n) \smallsetminus \operatorname{Var}(\xi_\tau)$ defined by

$$\mathbb{C}[\xi_\tau^{-1}] \otimes_{\mathbb{C}[\xi_\tau]} \left(\langle \xi_i \mid i \notin \tau \rangle + \langle \xi^u - \xi^v \mid u, v \in \mathbb{N}^n, u_i = v_i = 0 \text{ for } i \notin \tau, Au = Av \rangle + \langle Ax\xi \rangle \right). \tag{2.4}$$

Note that the polynomials in (2.4) can be viewed as to not involve the variables x_i for $i \notin \tau$; in particular, if $(\overline{x}, \overline{\xi}) \in \overline{C_\tau}$, then $(\mathbb{C}^{\bar{\tau}} \times \overline{x}_\tau) \times \{\overline{\xi}\} \subseteq \overline{C_\tau}$. Here $\mathbb{C}^{\bar{\tau}}$ denotes the affine subspace of X whose coordinates indexed by τ are zero, and $\overline{x}_\tau$ is the point in $\mathbb{C}^\tau$ whose coordinates indexed by τ coincide with those of $\overline{x}$.

Theorem 2.5. [SW08, Corollary 4.17] *The L-characteristic variety of $D/H_A(\beta)$ is*

$$\operatorname{Char}^L(D/H_A(\beta)) = \bigsqcup_{\tau \in \Phi(A,L)} C_\tau = \bigcup_{\tau \in \Phi(A,L)} \overline{C_\tau}. \qquad \square$$

The L-characteristic variety of a holonomic binomial D-module has been computed in [CF12]:

Theorem 2.6. [CF12, Theorem 4.3] *If the binomial D-module $M = D/(I, E_A - \beta)$ is holonomic, then the L-characteristic variety of M is the union of the L-characteristic varieties of the binomial modules $D/(\sqrt{\mathscr{C}}, E_A - \beta)$, where the union runs over the toral primary components $\mathscr{C}$ of I such that $-\beta \in \operatorname{qdeg}(\mathbb{C}[\partial]/\mathscr{C})$.*

3. The L-holonomicity of binomial D-modules

Using some ideas from [SW08], we prove here our first main result:

Theorem 3.1. *Let M be a binomial D-module.*

1. *The module M is holonomic if and only if M is L-holonomic for some (equivalently, every) projective weight vector L on D.*
2. *Furthermore, the module M is not holonomic if and only if $\operatorname{Char}^L(M)$ has a component in $T^*(\mathbb{C}^*)^n$ of dimension greater than n for some (equivalently, every) projective weight vector L on D.*

Note that [SST00, Theorem 1.4.12] uses a Gröbner walk argument to show equivalence of holonomicity and L-holonomicity for any cyclic module D/I, but with different assumptions on L

than we make here: [SST00] requires that all coordinates of L be nonnegative and $L_x + L_\partial > 0$ coordinatewise, while we ask for projective weight vectors whose coordinate sums are positive, but whose individual entries may be negative.

Remark 3.2. Theorem 3.1(2) can be restated as "A binomial D-module M is L-holonomic if and only if its restriction to $(\mathbb{C}^*)^n$ is L-holonomic." In this form, it is easy to see the usefulness of this result for the purposes of the companion article [BMW13]. In that paper, binomial D-modules are related to classical hypergeometric systems using a specially constructed functor that preserves important D-module theoretic properties. Part of that functor is restriction to the ambient torus $(\mathbb{C}^*)^n$. Theorem 3.1.(2) ensures that, when applied to binomial D-modules, the functor constructed in [BMW13] preserves L-holonomicity attributes of the original module, which in turn provides long-sought results about the holonomicity of classical hypergeometric systems of differential equations. ⬡

Example 3.3. On $X = \mathbb{C}^2$, the left ideal $\langle x_1^2\partial_1, x_1\partial_2\rangle$ fails to be L-holonomic for all projective weight L as its L-characteristic variety contains the hyperplane given by $x_1 = 0$. However, its restriction to $(\mathbb{C}^*)^2$ is the ideal generated by ∂_1 and ∂_2, which is clearly L-holonomic for all projective weight vectors L. ⬡

Notation 3.4. For A as in Convention 2.1, let $\breve{A} \in \mathbb{Z}^{k\times n}$ denote a matrix of full rank k with $d < k < n$, and assume that A is the submatrix of $\breve{A}$ consisting of its first d rows. Let $L \in \mathbb{Q}^n \times \mathbb{Q}^n$ be a projective weight vector.

Given a face $\tau \in \Phi(\breve{A}, L)$, put $\partial_\tau = \prod_{i\in\tau}\partial_i$ and

$$\breve{C}_\tau := \{(x,\xi) \mid \xi_i \neq 0 \text{ for all } i \in \tau\} \cap \mathrm{Var}(\langle \xi_i \mid i \notin \tau\rangle + \mathrm{gr}^L(I_{\breve{A}}) + \mathrm{gr}^L(E_A - \beta)).$$

⬡

Recall from [SW08] that $\mathrm{gr}^L(I_{\breve{A}}) \subseteq \mathrm{gr}^L(D)$ has a minimal component for every $\tau \in \Phi(\breve{A}, L)$ of dimension $k-1$ (a *facet*). Since $I_{\breve{A}} \subseteq \mathbb{C}[\partial]$, we have abused notation and written $\mathrm{gr}^L(I_{\breve{A}})$ in place of $\mathrm{gr}^L(D \cdot I_{\breve{A}})$.

Proposition 3.5. *Let $\breve{A}$ be as in Notation 3.4. Then for any facet $\tau \in \Phi(\breve{A}, L)$, $\breve{C}_\tau$ is contained in* $\mathrm{Char}^L(D/(I_{\breve{A}}, E_A - \beta))$.

Proof. Fix a facet $\tau \in \Phi(\breve{A}, L)$; then the matrix τ has full rank k. We argue as in [SW08, Theorem 3.10]. Since $\mathrm{gr}^L(E_{\breve{A}})$ is a regular sequence on $\mathrm{gr}^L(D[\partial_\tau^{-1}]/I_{\breve{A}})$, so is $\mathrm{gr}^L(E_A)$. The spectral sequence of a filtered complex shows that

$$\mathrm{gr}^L\left(\frac{D[\partial_\tau^{-1}]}{(I_{\breve{A}}, E_A - \beta)}\right) = \frac{\mathrm{gr}^L(D[\partial_\tau^{-1}])}{\mathrm{gr}^L(I_{\breve{A}}) + \mathrm{gr}^L(E_A - \beta)}.$$

(A similar argument is made in in [SST00, Theorem 4.3.5].) However,

$$\mathrm{gr}^L(I_{\breve{A}}) + \mathrm{gr}^L(E_A - \beta) \subseteq \langle \xi_i \mid i \notin \tau\rangle + \mathrm{gr}^L(I_{\breve{A}}) + \mathrm{gr}^L(E_A - \beta),$$

and localizing at ∂_τ we conclude that $\breve{C}_\tau$ is contained in $\mathrm{Char}^L(D/(I_{\breve{A}}, E_A - \beta))$. □

Proposition 3.6. *If $\breve{A}$ is as in Notation 3.4, then $D/(I_{\breve{A}}, E_A - \beta)$ fails to be L-holonomic for all projective weight vectors L.*

Proof. Fix a projective weight vector L. By Proposition 3.5, $\breve{C}_\tau \subseteq \operatorname{Char}^L(D/(I_{\breve{A}}, E_A - \beta))$ for each face $\tau \in \Phi(\breve{A}, L)$ of dimension $k-1$, and $\operatorname{gr}^L(E_A - \beta)$ is a regular sequence on $\operatorname{gr}^L(D[\partial_\tau^{-1}]/I_{\breve{A}})$. Thus, $\dim(\breve{C}_\tau) = n + k - d > n$ and $D/(I_{\breve{A}}, E_A - \beta)$ is not L-holonomic. □

Proof of Theorem 3.1.(1). Let $M = D/(I, E - \beta)$ be a binomial D-module. If M is holonomic, then it is L-holonomic for all projective weight vectors L since all A-hypergeometric D-modules (such as the modules $D/(\sqrt{\mathscr{C}}, E_A - \beta)$ for $\mathscr{C}$ toral) are L-holonomic.

Now assume that M is a non-holonomic binomial D-module. The associated primes of I are of the form $\mathfrak{p} = \mathbb{C}[\partial] \cdot (I_0 + \langle \partial_i \mid i \notin \sigma \rangle)$, where $\sigma \subseteq \{1, \ldots, n\}$, I_0 is generated by binomials in $\mathbb{C}[\partial_j \mid j \in \sigma] =: \mathbb{C}[\partial_\sigma]$, and $I_0 \cap \mathbb{C}[\partial_\sigma]$ is isomorphic to a toric ideal after rescaling the variables [ES96, Corollary 2.6].

By [DMM10b, Theorems 5.6, 6.3] there exists a primary component $\mathscr{C}$ of I corresponding to an Andean associated prime $\mathfrak{p}$ such that $-\beta \in \operatorname{qdeg}(\mathbb{C}[\partial]/\mathscr{C})$ and such that $D/(\mathscr{C}, E_A - \beta)$ is not holonomic. The argument in the proof of [DMM10b, Theorem 5.6] allows us to reduce to the case when $-\beta \in \operatorname{qdeg}(\mathbb{C}[\partial]/\mathfrak{p})$. In this case, the Andean condition ensures that $D/(\mathfrak{p}, E_A - \beta)$ is (after rescaling of the variables) a binomial D-module as in Notation 3.4. Thus, the proof is complete by Proposition 3.6. □

We now prove a stronger version of Proposition 3.6.

Proposition 3.7. *If $\breve{A}$ is as in Notation 3.4, then $\operatorname{Char}^L(D/(I_{\breve{A}}, E_A - \beta))$ has a component in $T^*(\mathbb{C}^*)^n$ of dimension $n + k - d$.*

Proof. Let $\breve{a}_i$ denote the ith column of $\breve{A}$ and let N be an $\breve{A}$-graded $\mathbb{C}[\partial]$-module where $\deg(x_i) = \breve{a}_i = -\deg(\partial_i)$. Fix some $\beta_{\breve{A}} \in \mathbb{C}^k$ that agrees with β in its first d coordinates, and let $\mathcal{K}_\bullet(E_{\breve{A}} - \beta_{\breve{A}}; N)$ and $\mathcal{H}_\bullet(E_{\breve{A}} - \beta_{\breve{A}}; N)$ respectively denote the Euler–Koszul complex and its homology in the sense of [MMW05, SW08].

Recall that the L-initial terms of x_j and ∂_j are denoted by x_j and ξ_j respectively. Let τ be a facet of $\Phi(\breve{A}, L)$, and set $\xi_\tau := \prod_{j\in\tau} \xi_j$, $\partial_\tau := \prod_{j\in\tau} \partial_j$ and let $I_\tau \subseteq \mathbb{C}[\partial_j \mid j \in \tau]$ be the toric ideal defined by (the submatrix of $\breve{A}$ whose columns are indexed by) τ.

By [SW08], the spectral sequence

$$H_\bullet(\operatorname{gr}^L(D[\partial_\tau^{-1}] \otimes_D \mathcal{K}_\bullet(E_{\breve{A}} - \beta_{\breve{A}}; D/I_{\breve{A}}))) \Rightarrow \operatorname{gr}^L(D)[\xi_\tau^{-1}] \otimes_{\operatorname{gr}^L(D)} \operatorname{gr}^L(\mathcal{H}_\bullet(E_{\breve{A}} - \beta_{\breve{A}}, D/I_{\breve{A}}))$$

induced by the L-filtration on the localized Euler–Koszul complex to $D/I_{\breve{A}}$ collapses, essentially since $\operatorname{gr}^L(E_{\breve{A}} - \beta_{\breve{A}})$ forms a regular sequence on $\operatorname{gr}^L(D)[\xi_\tau^{-1}]/\left(\operatorname{gr}^L(D)[\xi_\tau^{-1}] \cdot \operatorname{gr}^L(I_{\breve{A}})\right)$.

The facet τ may or may not be a pyramid in the sense of [SW12, Definition 2.4]. By Remark 2.5 in loc. cit. that there is a unique face σ of τ such that τ is a pyramid over σ and σ is not a pyramid. In particular, slightly abusing notation, $I_\tau = I_\sigma$ and I_τ-primary ideals are I_σ-primary ideals. Hence

$$H_\bullet(\operatorname{gr}^L(D[\partial_\sigma^{-1}] \otimes_D \mathcal{K}_\bullet(E_{\breve{A}} - \beta_{\breve{A}}; D/I_{\breve{A}}))) \Rightarrow \operatorname{gr}^L(D)[\xi_\sigma^{-1}] \otimes_{\operatorname{gr}^L(D)} \operatorname{gr}^L(\mathcal{H}_\bullet(E_{\breve{A}} - \beta_{\breve{A}}, D/I_{\breve{A}}))$$

also collapses because $\operatorname{gr}^L(E_{\breve{A}})$ is a regular sequence on $\operatorname{gr}^L(D)[\xi_\sigma^{-1}]/\left(\operatorname{gr}^L(D)[\xi_\sigma^{-1}] \cdot \operatorname{gr}^L(I_{\breve{A}})\right)$. In particular, $\operatorname{gr}^L(D)[\xi_\sigma^{-1}] \otimes_{\operatorname{gr}^L(D)} \operatorname{gr}^L(D/H_{\breve{A}}(\beta_{\breve{A}})) = \operatorname{gr}^L(D)[\xi_\sigma^{-1}]/\left(\operatorname{gr}^L(D)[\xi_\sigma^{-1}] \cdot (P_\tau + \operatorname{gr}^L(E_{\breve{A}}))\right)$, where P_τ is the $\left(\operatorname{gr}^L(I_\tau) + \langle \xi_i \mid i \notin \tau \rangle\right)$-primary component of $\operatorname{gr}^L(I_{\breve{A}})$.

It follows that in $\mathrm{gr}^L(D)[\xi_\sigma^{-1}]$, $\mathrm{gr}^L(E_A)$ is regular on $\mathrm{gr}^L(I_{\breve{A}})$ and

$$\mathrm{gr}^L(D)[\xi_\sigma^{-1}] \otimes_{\mathrm{gr}^L(D)} \mathrm{gr}^L(D/(I_{\breve{A}}, E_A - \beta) = \mathrm{gr}^L(D)[\xi_\sigma^{-1}]/\left(\mathrm{gr}^L(D)[\xi_\sigma^{-1}] \cdot (P_\tau + \mathrm{gr}^L(E_A))\right).$$

We now show that the variety of $(P_\tau + \mathrm{gr}^L(E_A)) \subseteq \mathrm{gr}^L(D)[\xi_\sigma^{-1}]$ has a component of dimension at least $n+1$ that meets $T^*(\mathbb{C}^*)^n$.

The regularity of $\mathrm{gr}^L(E_A)$ on $\mathrm{gr}^L(I_{\breve{A}})$ in the localized ring ensures that every component of $\mathrm{gr}^L[\xi_\sigma^{-1}](P_\tau + \mathrm{gr}^L(E_A))$ has dimension $2n - d - (n-d) = n + k - d$, as τ is a facet and so the height of P_τ is $n - d$. It remains to be shown that one of its components meets $T^*(\mathbb{C}^*)^n$. We may safely replace P_τ by $\sqrt{P_\tau} = \mathrm{gr}^L(I_\tau) + \langle \xi_i \mid i \notin \tau\rangle = \mathrm{gr}^L(I_\sigma) + \langle \xi_i \mid i \notin \tau\rangle$. Then we have

$$\mathrm{gr}^L(D)[\xi_\sigma^{-1}]\big(\mathrm{gr}^L(I_\sigma)+\langle \xi_i \mid i \notin \tau\rangle+\mathrm{gr}^L(E_A)\big) = \mathrm{gr}^L(D)[\xi_\sigma^{-1}]\big(\mathrm{gr}^L(I_\sigma)+\langle \xi_i \mid i \notin \tau\rangle+\mathrm{gr}^L(E_{A,\tau})\big),$$

where $E_{A,\tau}$ denotes the Euler operators arising from the submatrix of A whose columns are indexed by τ. As $\tau \supseteq \sigma$ is a pyramid, the right hand side above is equal to

$$\mathrm{gr}^L(D)[\xi_\sigma{}^{-1}]\left(\mathrm{gr}^L(I_\sigma) + \langle \xi_i \mid i \notin \tau\rangle + \mathrm{gr}^L(E_{A,\sigma}) + \langle x_i\xi_i \mid i \in \tau \smallsetminus \sigma\rangle\right),$$

and, in particular, it is contained in the ideal $\mathrm{gr}^L(D)[\xi_\sigma^{-1}]\left(\mathrm{gr}^L(I_\sigma) + \langle \xi_i \mid i \notin \sigma\rangle + \mathrm{gr}^L(E_{A,\sigma})\right)$ of height $n - d + k$.

To avoid confusion, for $\sigma \in \Phi(\breve{A}, L)$, we denote by $C_{\sigma,\breve{A}}$ the conormal space to the orbit of the point $\mathbf{1}_\sigma \in \mathbb{C}^n$ under the $(\mathbb{C}^*)^k$-action (defined by $\breve{A}$). Previously defined for $\tau \in \Phi(A, L)$ and the corresponding $(\mathbb{C}^*)^d$-action, such a conormal was written C_τ, see (2.4).

Suppose now that $\mathrm{Var}\left(\mathrm{gr}^L(D)[\xi_\sigma^{-1}]\left(\mathrm{gr}^L(I_\sigma) + \langle \xi_i \mid i \notin \sigma\rangle + \mathrm{gr}^L(E_{A,\sigma})\right)\right)$ does not meet $T^*(\mathbb{C}^*)^n$. As this variety contains the conormal $C_{\sigma,\breve{A}}$, none of its components containing $C_{\sigma,\breve{A}}$ can meet $T^*(\mathbb{C}^*)^n$ in $\xi_\sigma \neq 0$. Any such component must be contained in a hyperplane $\mathrm{Var}(x_i)$ inside the cotangent space of $\mathbb{C}^n \smallsetminus \mathrm{Var}(\xi_\sigma)$. In particular, this must then hold for $C_{\sigma,\breve{A}}$ itself because its generic point is over $\xi_\sigma \neq 0$. We show now that this is impossible.

Let $\mathrm{Var}(x_i)$ be the presumed hyperplane that contains $C_{\sigma,\breve{A}}$; since for the $(\mathbb{C}^*)^k$-orbit of $\mathbf{1}_\sigma$ the variable x_i is a cotangent variable, this implies that the toric ideal I_σ does not involve ∂_i. In turn, σ must be a pyramid by [SW12, Remark 2.5]. However, this contradicts our choices, so $C_{\sigma,\breve{A}}$ must meet $T^*(\mathbb{C}^*)^n$. □

It would be interesting to determine the structure of $\mathrm{Char}^L(D/(I_{\breve{A}}, E_A - \beta))$. Frequently, different facets of the L-umbrella of A give rise to the same "bad" component in the proof of Proposition 3.7.

Proof of Theorem 3.1.(2). Fix a projective weight vector L on D. By Theorem 3.1.(1), we may replace "holonomic" by "L-holonomic." The "if" direction is clear from the definition of L-characteristic variety. For the "only if" direction, as in the proof of Theorem 3.1.(1) above, we can reduce to the case of M as in Proposition 3.7. □

4. Finite rank, singular loci, and holonomicity

4.1. **Properness of the singular locus.** As shown by Kashiwara, holonomicity implies finite rank for any D-module (see [SST00, Proposition 1.4.9]). The converse need not hold as obvious cases such as Example 4.3 show. Interestingly, for an arbitrary cyclic module D/I, the finiteness of the rank implications on the structure of the singular locus:

Proposition 4.1. *If I is a left D-ideal, then D/I is of finite rank if and only if* $\operatorname{Sing}(D/I)$ *is a proper subset of X.*

Proof. By [SST00, Corollary 1.4.14], the rank of D/I is the length of $\operatorname{gr}^F(D/I)_{\langle\xi\rangle}$ over $\operatorname{gr}^F(D)_{\langle\xi\rangle}$. Thus, $\operatorname{rank}(D/I) < \infty$ is equivalent to $\operatorname{gr}^F(D/I)_{\langle\xi\rangle}$ being an Artinian $\operatorname{gr}^F(D)_{\langle\xi\rangle}$-module. To put this another way, for all (minimal) primes $\mathfrak{p}$ of $\operatorname{gr}^F(I)$, the ring $(\operatorname{gr}^F(D)/\mathfrak{p})_{\langle\xi\rangle}$ is Artinian. Equivalently, for all (minimal) primes $\mathfrak{p}$ of $\operatorname{gr}^F(I)$, $\mathfrak{p}$ is not properly contained in $\langle\xi\rangle$; in other words, such a $\mathfrak{p}$ either equals $\langle\xi\rangle$ or contains an element in $\operatorname{gr}^F(D)$ that lies outside of $\langle\xi\rangle$. This means exactly that each (minimal) prime of $\operatorname{gr}^F(I)$ that is different from $\langle\xi\rangle$ contains a nontrivial polynomial in x. This is because $\operatorname{gr}^F(I)$, and hence each of its (minimal) primes, is ξ-homogeneous and $\langle\xi\rangle$ contains all elements of positive degree. Therefore, the rank of D/I is finite exactly when $\operatorname{gr}^F(I) : \langle\xi\rangle^\infty$ contains a polynomial in x, which is equivalent to properness of the singular locus of D/I, as desired. □

For an arbitrary left D-ideal I, the holonomicity of D/I implies the properness of its singular locus [Bjö79, Subsection 5.4.6]. That the converse holds for binomial D-modules is a special feature of this class:

Theorem 4.2. *A binomial D-module M is holonomic if and only if its singular locus* $\operatorname{Sing}(M)$ *is a proper subset of X.*

Example 4.3. Over $X = \mathbb{C}^2$, the module $D/\langle x_1\rangle$ is not holonomic because its characteristic variety is a hypersurface in $\mathbb{C}^4$. However, $\operatorname{Sing}(D/\langle x_1\rangle) = \operatorname{Var}(x_1)$, and as such, it is a proper subset of X. For other examples of non-holonomic modules with proper singular loci, see Example 4.4 and [SST00, Example 1.4.10]. ○

Proof of Theorem 4.2. For binomial D-modules, holonomicity is equivalent to having finite rank by [DMM10b, Theorem 6.3]. Thus, the result follows by combining [SST00, Proposition 1.4.9], [DMM10b, Theorem 6.3], and our Proposition 4.1. □

Theorem 4.2 was inspired by [PST05, Proof of Theorem 7], where Proposition 4.1 was used for Horn hypergeometric systems. Horn hypergeometric systems are closely related to certain binomial D-modules (see [BMW13]); however, even for these systems, properness of the singular locus is not equivalent to holonomicity, as shown in the following example.

Example 4.4. On $X = \mathbb{C}^3$, let $\theta_i := x_i\partial_i$ for $1 \leq i \leq 3$. The left D-ideal

$$\begin{aligned} I = D \cdot \langle (\theta_1 + 2\theta_2 + \theta_3 + 2)\theta_1 - x_1(\theta_1 + 2\theta_2)\theta_1, \\ (\theta_1 + 2\theta_2 + \theta_3 + 2)(\theta_1 + 2\theta_2 + \theta_3 + 1)\theta_2 + x_2(\theta_1 + 2\theta_2)(\theta_1 + 2\theta_2 + 1)\theta_2, \\ (\theta_1 + 2\theta_2 + \theta_3 + 2) + x_3\theta_3 \rangle \end{aligned}$$

is a nonconfluent Horn hypergeometric system of finite rank. However, D/I is not holonomic, as witnessed by the component $\operatorname{Var}(\langle x_3, x_1\xi_1 + x_2\xi_2\rangle)$ of its characteristic variety. On the other hand, computations in `Macaulay2` [M2] verify that $\operatorname{Sing}(D/I)$ is indeed a proper subset of X. ○

4.2. **A formula for the singular locus.** We finally produce now a polynomial that defines the divisorial singular locus of a binomial D-module. The point is the selection of the contributing toral ideals, see Corollary 4.6.

We first consider the divisorial singular locus of an A-hypergeometric system. By definition, the divisorial singular locus of $D/H_A(\beta)$ is obtained by removing the variety $\operatorname{Var}(\xi_1,\dots,\xi_n)$ from $\operatorname{Char}^F(D/H_A(\beta))$, projecting the resulting set onto X, and discarding the components of codimension two or higher.

A theorem, quoted below, of Gelfand, Kapranov, and Zelevinsky [GKZ88, GKZ89] as well as Adolphson [Ado94] describes the divisorial singular locus of an A-hypergeometric system. We shall use it to characterize the divisorial singular locus of a binomial D-module.

Let $f = \bar{x}_1 t^{a_1} + \cdots + \bar{x}_n t^{a_n}$. The Zariski closure of the set

$$\left\{ \bar{x} \in \mathbb{C}^n \;\middle|\; \exists\, \bar{t} \in (\mathbb{C}^*)^d \text{ such that } f(\bar{t}) = \frac{\partial f}{\partial t_i}(\bar{t}) = 0 \text{ for } i = 1,\dots,d \right\}$$

is called the *A-discriminantal variety*. If this (irreducible) variety is a hypersurface, its defining polynomial is called the *A-discriminant*, denoted Δ_A. If the codimension of the A-discriminantal variety is at least 2, then we set $\Delta_A = 1$. The *principal A-determinant*, denoted $\mathscr{E}_A$, is defined in [GKZ94, Chapter 10, Equation 1.1],

$$\mathscr{E}_A = \pm \prod_{\tau \text{ face of } \operatorname{conv}(A)} (\Delta_\tau)^{\mu(\tau)}$$

for certain positive integers $\mu(\tau)$. See [GKZ94, Chapter 10, Theorem 1.2] for more details, as well as [Kap91] for a parametric treatment of A-discriminants.

Theorem 4.5. *The divisorial singular locus of an A-hypergeometric system $D/H_A(\beta)$ is the zero set of the product of all τ-discriminants arsing from the faces τ of $\Phi(A,F)$. In particular, if all of the columns of A lie in a hyperplane off the origin, then* $\operatorname{Sing}^1(D/H_A(\beta)) = \operatorname{Var}(\mathscr{E}_A)$. □

Returning to the case of a binomial D-module $M = D/(I, E_A - \beta)$, recall from Theorem 2.6 that the L-characteristic variety of M is the union of L-characteristic varieties of A-hypergeometric systems (up to a rescaling of the variables) given by certain toral associated primes of I. An alternative characterization for the toral components of I is that $\mathscr{C}$ is toral if and only if $D \cdot (\sqrt{\mathscr{C}}, E_A - \beta)$ is isomorphic to an $\tilde{A}$-hypergeometric system by rescaling the variables, where $\tilde{A}$ is the submatrix of A consisting of the columns a_i such that $\partial_i \notin \sqrt{\mathscr{C}}$ [DMM10a, Corollary 4.8].

This observation provides a description of the characteristic variety of a binomial D-module in terms of those of such $\tilde{A}$-hypergeometric modules. Consequently, the divisorial singular locus of a holonomic binomial D-module can be expressed in terms of principal τ-determinants.

Corollary 4.6. *The singular locus of a holonomic binomial D-module $D/(I, E_A - \beta)$ is the union of $\operatorname{Sing}(D/(\sqrt{\mathscr{C}}, E_A - \beta))$, where the union runs over the toral primary components $\mathscr{C}$ of I such that $-\beta \in \operatorname{qdeg}(\mathbb{C}[\partial]/\mathscr{C})$. The divisorial singular locus of $D/(I, E_A - \beta)$ is a union of rescaled discriminantal varieties, given by the product of the rescaled τ-discriminants for $\tau \in \Phi(A_{\mathfrak{p}}, L)$, where, for a toral associated prime $\mathfrak{p}$ of I, the matrix $A_{\mathfrak{p}}$ has columns a_i whenever $\partial_i \notin \mathfrak{p}$.*

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