

Bockstein maps and local cohomology

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Bockstein Setup

R commutative Noetherian ring

M R -module

p nzdv on M

$F: R\text{-mods} \rightarrow R\text{-mods}$ covariant additive R -linear δ -functor.

Remark

More general setups are possible, including the case of singular cohomology with coefficients in the R -module M .

Needed is an endomorphism of a complex.

Bockstein definition

From

$$0 \rightarrow M \xrightarrow{p} M \rightarrow M/pM \rightarrow 0$$

we obtain

$$F^k(M/pM) \xrightarrow{\delta_p^k} F^{k+1}(M) \xrightarrow{p} F^{k+1}(M) \xrightarrow{\pi_p^{k+1}} F^{k+1}(M/pM).$$

The *Bockstein map* $\beta_p^k = \beta_p^k(F, M)$ is

$$\pi_p^{k+1} \circ \delta_p^k: F^k(M/pM) \rightarrow F^{k+1}(M/pM).$$

A cohomology operation.

Bockstein Examples

- $(R, p, F) = (\mathbb{Z}, p, H_{Sing}^\bullet)$.

Classical case, used to study lens spaces.

In 3D: $L(p, q) = \mathbb{S}^3 / \left(\begin{array}{cc} \omega & 0 \\ 0 & \omega^q \end{array} \right), \omega^p = 1$. (Tietze 1908)

Fact: $L(5, 1), L(5, 2)$ have equal π_1 and H_* , but are not homotopy equivalent.

Fact: $L(7, 1), L(7, 2)$ homotopy equivalent but not homeomorphic.

- $(R, p, F) = (R, f, H^\bullet(f; -))$.

$$H^0(f; R/pR) = (0 :_{R/pR} f), \quad H^1(f; R/pR) = R/(p, f)R,$$

$$\beta(r \bmod(p)) = \frac{fr}{p} \bmod(p, f).$$

- F=Local cohomology (next).

Definition of local cohomology

Setup

$R =$ commutative ring,
 $(f_1, \dots, f_t) R = I \subseteq R$ an ideal,
 M an R -module.

Consider $\check{C}_i^\bullet = (R \rightarrow R[f_i^{-1}])$ and

$$\check{C}_f^\bullet = \bigotimes_1^t \check{C}_i^\bullet$$

Definition

$$H_i^k(M) := H^k(M \otimes \check{C}_f^\bullet).$$

Why care about local cohomology?

- If $\sqrt{I} = \sqrt{J}$ then $H_I^\bullet = H_J^\bullet$ (“geometric”).
- $R = \mathbb{C}[x_1, \dots, x_n]$, $V = \text{Var}(I)$, $U = \mathbb{C}^n \setminus V$.

$$E_2^{p,q} = H_{deRham}^p(H_I^q(R)) \Rightarrow H_{sing}^{p+q+n-1}(U).$$

- (R, \mathfrak{m}) local: $\left[H_{\mathfrak{m}}^{< \dim(R)}(R) = 0 \right] \Leftrightarrow [R \text{ is CM}]$.
- Relations to hypergeometric systems, Riemann–Hilbert correspondence, tight closure, topology.

Stanley–Reisner theory

Setup

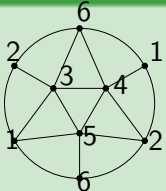
$R = \mathbb{C}[x_1, \dots, x_n]$, $\Delta \subseteq \Delta_n = n$ -simplex, $\mathfrak{m} = \mathbf{x}R$.

Definition

$$I_\Delta = \{\mathbf{x}^\sigma \mid \sigma \notin \Delta\}R, \quad S_\Delta = R/I_\Delta$$

(squarefree monomial) Stanley–Reisner ideal, and ring.

Example

$\Delta = \mathbb{RP}^2$:  Here, $I_{\mathbb{RP}^2} = (x_1x_2x_5, \dots, x_4x_5x_6)R$.

Stanley–Reisner, local cohomology, Bockstein

Theorem (Hochster, simplified)

For $0 \neq p \in \text{Spec } \mathbb{Z}$,

$$H_m^i(S_\Delta/pS_\Delta)_0 = \tilde{H}^{i-1}(\Delta, \mathbb{Z}/p\mathbb{Z}).$$

(Really, Hochster describes **all** graded pieces...)

Theorem (SW)

The Bockstein construction respects the decomposition of Hochster's theorem.

Moreover, the following are equivalent:

- *Bocksteins are zero on $H_m^\bullet(S_\Delta/pS_\Delta)$;*
- *Bocksteins are zero on $H_{I_\Delta}^\bullet(R/pR)$;*
- *Bocksteins are zero on certain subcomplexes of Δ .*

Bockstein and Stanley–Reisner

Example

A nonzero Bockstein in topology:

$$H_{\text{sing}}^1(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z}) = H_{\text{sing}}^2(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$$

Resolution of the Stanley–Reisner ideal:

$$0 \rightarrow R \rightarrow R^7 \rightarrow R^{15} \rightarrow R^{10} \rightarrow R \rightarrow R/I_{\Delta} \rightarrow 0.$$

Last map: $(x_1, \dots, x_6, 2)$.

Diagram chase: the **2** is responsible for nonzero

$$\beta_2: H_{I_{\Delta}}^3(R/2R) \rightarrow H_{I_{\Delta}}^4(R/2R)$$

via $(0, 0, 0, 0, 0, 1) \in \text{Ext}_{R/2R}^3(R/(2, I_{\Delta}), R/2R) \rightarrow$

$H_{I_{\Delta}}^3(R/2R) \rightarrow H_{I_{\Delta}}^4(R/2R) \leftarrow \text{Ext}_{R/2R}^3(R/(2, I_{\Delta}), R/2R) \ni (1).$

Finiteness of local cohomology

Setup

R Noeth., M fin. gen., I ideal.

- Grothendieck: \mathfrak{m} maximal $\Rightarrow H_{\mathfrak{m}}^{\bullet}(M)$ Artinian
- Huneke: $\text{Ass}(H_I^i(R))$ finite ?
- Huneke–Sharp, Lyubeznik: R regular (local) containing field then **yes**.
- Singh: In general, **no** (next slide).
- Lyubeznik: R mixed char., regular, local, unramified, then **yes**.
- Lyubeznik: R regular then $H_I^i(R)$ finite Ass ?
(open for $\mathbb{Z}[x_1, \dots, x_n]$!!!)

Example (Singh)

Setup

$R = \mathbb{Z}[x, y, z, u, v, w]/(xu + yv + zw)$, $I = (x, y, z)$,
 $0 < p \in \text{Spec } \mathbb{Z}$.

- $\eta_p = [(u/yz, -v/xz, w/xy)] \in H_1^2(R/pR)$.
- $\beta_p^2(\text{Frob}(\eta_p)) = \left[\frac{(ux)^p + (vy)^p + (wz)^p}{p(xyz)^p} \right] \neq 0$.
- Thus, $\delta: H_1^2(R/pR) \rightarrow H_1^3(R)$ sends $\text{Frob}(\eta_p)$ to $\delta(\text{Frob}(\eta_p)) \neq 0$.
- p kills $\text{Frob}(\eta_p)$, hence also $\delta(\text{Frob}(\eta_p))$.
- So $H_1^3(R)$ can't have finite Ass.

Main Theorem

Theorem

$R = \mathbb{Z}[x_1, \dots, x_n]$, $I = (f_1, \dots, f_t)R$, p *nzdv* on $H^{k+1}(f; R)$.

Then

$$\beta_p^k: H_I^k(R/pR) \rightarrow H_I^{k+1}(R/pR)$$

is zero.

Remark

$$\left| \bigcup_{k \in \mathbb{N}} \text{Ass}(H^{k+1}(f; R)) \right| < \infty.$$

Counterexamples to Lyubeznik can't be found with Singh's method!

Idea of proof

- $H_j^k(R) = \varinjlim_{i \in \mathbb{N}} H^k(f^i; R)$
- Let p nzdv on $H^{k+1}(f, R)$.
- $\psi: R \rightarrow R$, $\psi_p(x_i) = x_i^p$ is **flat**.
- $\Psi: R\text{-mods} \rightarrow R\text{-mods}$, $\Psi(M) = M \otimes_R \psi R$ **exact**.
- $\Psi^e \left(H^{k+1}(f; R) \xrightarrow{p} H^{k+1}(f; R) \right)$ **injective**.
- Bockstein to $\Psi^e(f)$ is **zero**.
- Mod p , $\Psi(f) = f^p$, prove comparison for Bocksteins, done.

Thank you!