SURVEY ON THE D-MODULE f^s

ULI WALTHER

WITH AN APPENDIX BY ANTON LEYKIN

Contents

In this survey we discuss various aspects of the singularity invariants with differential origin derived from the D-module generated by f^s. We should like to point the reader to some other works: [193] for V-filtration, Bernstein–Sato polynomials, multiplier ideals; [49] for all these and Milnor fibers; [216] and [161] for homogeneity and free divisors; [208] on details of arrangements, specifically their Milnor fibers, although less focused on D-modules.

We are greatly indebted to Nero Budur, Francisco J. Castro-Jiménez, Luis Narváez-Macarro, Wim Veys and an unknown referee for their careful reading of early versions of this article and their relentless hunt for errors. We claim ownership, and apologize for, all surviving mistakes, oversights and omissions.

1. Introduction

Notation 1.1. In this article, X will denote a complex manifold. Unless indicated otherwise, X will be $\mathbb{C}^n$.

Throughout, let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials in n variables over the complex numbers. We denote by $D = R\langle \partial_1, \ldots, \partial_n \rangle$ the Weyl algebra.

Key words and phrases. Bernstein–Sato polynomial, b-function, hyperplane, arrangement, zeta function, logarithmic comparison theorem, multiplier ideal, Milnor fiber, algorithmic, free.

This material is based in part upon work supported by the National Science Foundation under Grant No. 0932078 000, while UW was in residence at the Mathematical Science Research Institute (MSRI) in Berkeley, California, during the spring of 2013.

UW was supported in part by the NSF under grant DMS 0901123.

AL was supported in part by the NSF under grant DMS 1151297.

In particular, ∂_i denotes the partial differentiation operator with respect to x_i. If X is a general manifold, $\mathscr{O}_X$ (the sheaf of regular functions) and $\mathscr{D}_X$ (the sheaf of $\mathbb{C}$-linear differential operators on $\mathscr{O}_X$) take the places of R and D.

If $X = \mathbb{C}^n$ we use Roman letters to denote rings and modules; in the general case we use calligraphic letters to denote corresponding sheaves.

By the ideal J_f we mean the $\mathscr{O}_X$-ideal generated by the partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$; this ideal varies with the choice of coordinate system in which we calculate. In contrast, the Jacobian ideal $\operatorname{Jac}(f) = J_f + (f)$ is independent.

The ring D (resp. the sheaf $\mathscr{D}_X$) is coherent, and both left- and right-Noetherian; it has only trivial two-sided ideals [26, Thm. 1.2.5]. Introductions to the theory of D-modules as we use them here can be found in [120, 24, 26, 25].

The ring D admits the order filtration induced by the weight $x_i \to 0$, $\partial_i \to 1$. The order filtration (and other good filtrations) leads to graded objects $\operatorname{gr}_{(0,1)}(-)$, see [199]. The graded objects obtained from ideals are ideals in the polynomial ring $\mathbb{C}[x, \xi]$, homogeneous in the symbols of the differentiation operators; their radicals are closed under the Poisson bracket, and thus the corresponding varieties are involutive [116, 121]. For a D-module M and a component C of the support of $\operatorname{gr}_{(0,1)}(M)$, attach to the pair (M, C) the multiplicity $\mu(M, C)$ of $\operatorname{gr}_{(0,1)}(M)$ along C. The characteristic cycle of M is $\operatorname{charC}(M) = \sum_C \mu(M, C) \cdot C$, an element of the Chow ring on $T^*\mathbb{C}^n$. The module is *holonomic* if it is zero or if its characteristic variety is of dimension n, the minimal possible value.

Throughout, f will be a regular function on X, with divisor $\operatorname{Var}(f)$. We distinguish several homogeneity conditions on f:

- f is *locally (strongly) Euler-homogeneous* if for all $p \in \operatorname{Var}(f)$ there is a vector field θ_p defined near p with $\theta_p \bullet (f) = f$ (and θ_p vanishes at p).
- f is *locally (weakly) quasi-homogeneous* if near all $p \in \operatorname{Var}(f)$ there is a local coordinate system $\{x_i\}$ and a positive (resp. non-negative) weight vector $a = \{a_1, \ldots, a_n\}$ with respect to which $f = \sum_{i=1}^n a_i x_i \partial_i(f)$.
- We reserve *homogeneous* and *quasi-homogeneous* for the case when $X = \mathbb{C}^n$ and f is globally homogeneous or quasi-homogeneous.

To any non-constant $f \in R$, one can attach several invariants that measure the singularity structure of the hypersurface $f = 0$. In this article, we are primarily interested in those derived from the (parametric) annihilator $\operatorname{ann}_{D[s]}(f^s)$ of f^s:

Definition 1.2. Let s be a new variable, and denote by $R_f[s] \cdot f^s$ the free module generated by f^s over the localized ring $R_f[s] = R[f^{-1}, s]$. Via the chain rule

$$\partial_i \bullet (\frac{g}{f^k} f^s) = \partial_i \bullet (\frac{g}{f^k}) f^s + \frac{sg}{f^{k+1}} \cdot \frac{\partial f}{\partial x_i} f^s \tag{1.1}$$

for each $g(x, s) \in R[s]$, $R_f[s] \cdot f^s$ acquires the structure of a left $D[s]$-module. Denote by

$$\operatorname{ann}_{D[s]}(f^s) = \{P \in D[s] \mid P \bullet f^s = 0\}$$

the *parametric annihilator*, and by

$$\mathscr{M}_f(s) = D[s] / \operatorname{ann}_{D[s]}(f^s)$$

the cyclic $D[s]$-module generated by $1 \cdot f^s \in R_f[s] \cdot f^s$.

Bernstein's functional equation [23] asserts the existence of a differential operator $P(x,\partial,s)$ and a nonzero polynomial $b_{f,P}(s)\in\mathbb{C}[s]$ such that

$$P(x,\partial,s)\bullet f^{s+1}=b_{f,P}(s)\cdot f^s, \tag{1.2}$$

i.e. the existence of the element $P\cdot f-b_{f,P}(s)\in\operatorname{ann}_{D[s]}(f^s)$. Bernstein's result implies that $D[s]\bullet f^s$ is D-coherent (while $R_f[s]f^s$ is not).

Definition 1.3. The monic generator of the ideal in $\mathbb{C}[s]$ generated by all $b_{f,P}(s)$ appearing in an equation (1.2) is the *Bernstein–Sato polynomial* $b_f(s)$. Denote $\rho_f\subseteq\mathbb{C}$ the set of roots of $b_f(s)$.

Note that the operator P in the functional equation is only determined up to $\operatorname{ann}_{D[s]}(f^s)$. See [25] for an elementary proof of the existence of $b_f(s)$. Alternative (and more general) proofs are given in [120]; see also [24, 151, 169].

The $\mathbb{C}[s]$-module $\mathscr{M}_f(s)/\mathscr{M}_f(s+1)$ is precisely annihilated by $b_f(s)$. It is an interesting problem to determine for any $q(s)\in\mathbb{C}[s]$ the ideals $\mathfrak{a}_{f,q(s)}=\{g\in R\mid q(s)gf^s\in D[s]\bullet f^{s+1}\}$ from [230]. By [146], $\mathfrak{a}_{f,s+1}=R\cap(\operatorname{ann}_{D[s]}(f^s)+D[s]\cdot(f,J_f))$.

Question 1.4. *Is* $\mathfrak{a}_{f,s+1}=J_f+(f)$*?*

A positive answer would throw light on connections between $b_f(s)$ and cohomology of Milnor fibers.

Remark 1.5. At the 1954 International Congress of Mathematics in Amsterdam, I.M. Gel'fand asked the following question. Given a real analytic function $f\colon\mathbb{R}^n\to\mathbb{R}$, the assignment ($s\in\mathbb{C}$)

$$f(x)^s_+=\begin{cases} f(x)^s & \text{if } f(x)>0,\\ 0 & \text{if } f(x)\le 0\end{cases}$$

is continuous in x and analytic in s where the real part of s is positive. Can one analytically continue $f(x)^s_+$? Sato introduced $b_f(s)$ in order to answer Gel'fand's question; Bernstein [23] established their existence in general.

Remark 1.6. Let $m\in M$ be a nonzero section of a holonomic D-module. Generalizing the case $1\in R$ there is a functional equation

$$P(x,\partial,s)\bullet(mf^{s+1})=b_{f,P;m}(s)\cdot mf^s$$

with $b_{f,P;m}(s)\in\mathbb{C}[s]$ nonzero. The monic generator of the ideal $\{b_{f,P;m}(s)\}$ is the b-function $b_{f;m}(s)$, [117].

2. Parameters and numbers

For any complex number γ, the expression f^γ represents, locally outside $\operatorname{Var}(f)$, a multi-valued analytic function. Via the chain rule as in (1.1), the cyclic R_f-module $R_f\cdot f^\gamma$ becomes a left D-module, and we set

$$\mathscr{M}_f(\gamma)=D\bullet f^\lambda\cong D/\operatorname{ann}_D(f^\gamma).$$

There are natural $D[s]$-linear maps

$$\operatorname{ev}_f(\gamma)\colon\mathscr{M}_f(s)\to\mathscr{M}_f(\gamma),\qquad P(x,\partial,s)\bullet f^s\mapsto P(x,\partial,\gamma)\bullet f^\gamma,$$

and D-linear inclusions

$$\operatorname{inc}_f(s)\colon\mathscr{M}_f(s+1)\to\mathscr{M}_f(s),\qquad P(x,\partial,s)\bullet f^{s+1}\mapsto P(x,\partial,s)\cdot f\bullet f^s$$

with cokernel $\mathscr{N}_f(s) = \mathscr{M}_f(s)/\mathscr{M}_f(s+1) \cong D[s]/(\operatorname{ann}_{D[s]}(f^s) + D[s]f)$, and

$$\operatorname{inc}_f(\gamma)\colon \mathscr{M}_f(\gamma+1) \to \mathscr{M}_f(\gamma), \qquad P(x,\partial) \bullet f^{\lambda+1} \mapsto P(x,\partial)\cdot f \bullet f^{\lambda}$$

with cokernel $\mathscr{N}_f(\gamma) = \mathscr{M}_f(\gamma)/\mathscr{M}_f(\gamma+1) \cong D/(\operatorname{ann}_D(f^\gamma) + D\cdot f)$.

The kernel of the morphism $\operatorname{ev}_f(\gamma)$ contains the (two-sided) ideal $D[s](s-\gamma)$; the containment can be proper, for example if $\gamma = 0$. If $\{\gamma-1, \gamma-2, \dots\}$ is disjoint from the root set ρ_f then $\ker \operatorname{ev}_f(\gamma) = D[s]\cdot(s-\gamma)$, [117]. If $\gamma \notin \rho_f$ then $\operatorname{inc}_f(\gamma)$ is an isomorphism because of the functional equation; if $\gamma = -1$, or if $b_f(\gamma) = 0$ while ρ_f does not meet $\{\gamma-1, \gamma-2, \dots\}$ then $\operatorname{inc}_f(\gamma)$ is not surjective [230].

Question 2.1. *Does* $\operatorname{inc}_f(\gamma)$ *fail to be an isomorphism for all* $\gamma \in \rho_f$*?*

In contrast, the induced maps $\mathscr{M}_f(s)/(s-\gamma-1) \to \mathscr{M}_f(s)/(s-\gamma)$ are isomorphisms exactly when $\gamma \notin \rho_f$, [26, 6.3.15]. The morphism $\operatorname{inc}_f(s)$ is never surjective as $s+1$ divides $b_f(s)$. One sets

$$\tilde{b}_f(s) = \frac{b_f(s)}{s+1}.$$

By [217, 4.2], the following are equivalent for a section $m \neq 0$ of a holonomic module:

- the smallest integral root of $b_{f;m}(s)$ is at least $-\ell$;
- $(D \bullet m) \otimes_R R[f^{-1}]$ is generated by $m/f^\ell = m \otimes 1/f^\ell$;
- $(D \bullet m) \otimes_R R[f^{-1}]/D \bullet (m \otimes 1)$ is generated by m/f^ℓ;
- $D[s] \bullet mf^s \to (D \bullet m) \otimes_R R[f^{-1}]$, $P(s) \bullet (mf^s) \mapsto P(-\ell) \bullet (m/f^\ell)$ is an epimorphism with kernel $D[s]\cdot(s+\ell)mf^s$.

Definition 2.2. We say that f satisfies condition

- (A_1) (resp. (A_s)) if $\operatorname{ann}_D(1/f)$ (resp. $\operatorname{ann}_D(f^s)$) is generated by operators of order one;
- (B_1) if R_f is generated by $1/f$ over D.

Condition (A_1) implies (B_1) in any case [214]. Local Euler-homogeneity, (A_s) and (B_1) combined imply (A_1) [216], and for Koszul free divisors (see Definition 4.7 below) this implication can be reversed [214].

Condition (A_1) does not imply (A_s): $f = xy(x+y)(x+yz)$ is free (see Definition 4.1), and locally Euler-homogeneous and satisfies (A_1) and (B_1) [60, 61, 59, 67, 214], but $\operatorname{ann}_{D[s]}(f^s)$ and $\operatorname{ann}_D(f^s)$ require a second order generator.

Condition (A_1) implies local Euler-homogeneity if f has isolated singularities [213], or if it is Koszul-free or of the form $z^n - g(x,y)$ for reduced g [214]. In [73] it is shown that for certain locally weakly quasi-homogeneous free divisors $\operatorname{Var}(f)$, (A_1) holds for high powers of f, and even for f itself by [161, Rem. 1.7.4].

For an isolated singularity, f has (A_1) if and only if it has (B_1) and is quasi-homogeneous [213]. For example, a reduced plane curve (has automatically (B_1) and) has (A_1) if and only if it is quasi-homogeneous. See [201] for further results.

Condition (B_1) is equivalent to $\operatorname{inc}_f(-2), \operatorname{inc}_f(-3), \dots$ all being isomorphisms, and also to -1 being the only integral root of $b_f(s)$, [117]. Locally quasi-homogeneous free divisors satisfy condition (B_1) at any point, [66].

3. V-filtration and Bernstein–Sato polynomials

3.1. ***V*-filtration.** The articles [191, 145, 47, 49] are recommended for material on V-filtrations.

3.1.1. *Definition and basic properties.* Let Y be a smooth complex manifold (or variety), and let X be a closed submanifold (or -variety) of Y defined by the ideal sheaf $\mathscr{I}$. The V-filtration on $\mathscr{D}_Y$ along X is, for $k \in \mathbb{Z}$, given by

$$V^k(\mathscr{D}_Y) = \{P \in \mathscr{D}_Y \mid P \bullet \mathscr{I}^{k'} \subseteq \mathscr{I}^{k+k'} \quad \forall k' \in \mathbb{Z}\}$$

with the understanding that $\mathscr{I}^{k'} = \mathscr{O}_Y$ for $k' \leq 0$. The associated graded sheaf of rings $\mathrm{gr}_V(\mathscr{D}_Y)$ is isomorphic to the sheaf of rings of differential operators on the normal bundle $T_X(Y)$, algebraic in the fiber of the bundle.

Suppose that $Y = \mathbb{C}^n \times \mathbb{C}$ with coordinate function t on $\mathbb{C}$, and let X be the hyperplane $t = 0$. Then $V^k(D_Y)$ is spanned by $\{x^u \partial^v t^a \partial_t^b \mid a - b \geq k\}$. Given a coherent holonomic D_Y-module M with regular singularities in the sense of [122], Kashiwara and Malgrange [147, 114] define an exhaustive decreasing rationally indexed filtration on M that is compatible with the V-filtration on D_Y and has the following properties:

(1) each $V^\alpha(M)$ is coherent over $V^0(D_Y)$ and the set of α with nonzero $\mathrm{gr}_V^\alpha(M) = V^\alpha(M)/V^{>\alpha}(M)$ has no accumulation point;
(2) for $\alpha \gg 0$, $V^1(D_Y)V^\alpha(M) = V^{\alpha+1}(M)$;
(3) $t\partial_t - \alpha$ acts nilpotently on $\mathrm{gr}_V^\alpha(M)$.

The V-filtration is unique and can be defined in somewhat greater generality [47]. Of special interest is the following case considered in [147, 114].

Notation 3.1. Denote $R_{x,t}$ the polynomial ring $R[t]$, t a new indeterminate, and let $D_{x,t}$ be the corresponding Weyl algebra. Fix $f \in R$ and consider the regular $D_{x,t}$-module

$$\mathscr{B}_f = H^1_{f-t}(R[t]),$$

the unique local cohomology module of $R[t]$ supported in $f - t$. Then $\mathscr{B}_f$ is naturally isomorphic as $D_{x,t}$-module to the direct image (in the D-category) $i_+(R)$ of R under the graph embedding

$$i \colon X \to X \times \mathbb{C}, \qquad x \mapsto (x, f(x)).$$

Moreover, extending (1.1) via

$$t \bullet (g(x,s)f^{s-k}) = g(x,s+1)f^{s+1-k}; \qquad \partial_t \bullet (g(x,s)f^{s-k}) = -sg(x,s-1)f^{s-1-k},$$

the module $R_f[s] \otimes f^s$ becomes a $D_{x,t}$-module extending the $D[s]$-action where $-\partial_t t$ acts as s.

The existence of the V-filtration on $\mathscr{B}_f = i_+(R)$ is equivalent to the existence of generalized b-functions $b_{f;\eta}(s)$ in the sense of [117], see [118, 147]. In fact, one can recover one from the other:

$$V^\alpha(\mathscr{B}_f) = \{\eta \in \mathscr{B}_f \mid [b_{f;\eta}(-c) = 0] \Rightarrow [\alpha \leq c]\}$$

and the multiplicity of $b_{f;\eta}(s)$ at α is the degree of the minimal polynomial of $s - \alpha$ on $\mathrm{gr}_V^\alpha(D[s]\eta f^s / D[s]\eta f^{s+1})$, [182]. For more on this "microlocal approach" see [191].

3.2. **The log-canonical threshold.** By [125], see also [135, 235], the absolute value of the largest root of $b_f(s)$ is the *log-canonical threshold* $\mathrm{lct}(f)$ given by the supremum of all numbers s such that the local integrals

$$\int_{U \ni p} \frac{|dx|}{|f|^{2s}}$$

converge for all $p \in X$ and all small open U around p. Smaller lct corresponds to worse singularities; the best one can hope for is $\mathrm{lct}(f) = 1$ as one sees by looking at a smooth point. The notion goes back to Arnol'd, who called it (essentially) the complex singular index [10].

The point of *multiplier ideals* is to force the finiteness of the integral by allowing moderating functions in the integral:

$$\mathscr{I}(f, \lambda)_p = \{g \in \mathscr{O}_X \mid \frac{g}{f^\lambda} \text{ is } L^2\text{-integrable near } p \in \mathrm{Var}(f)\}$$

for $\lambda \in \mathbb{R}$. By [90], there is a finite collection of *jumping numbers* for f of rational numbers $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_\ell = 1$ such that $\mathscr{I}(f, \alpha)$ is constant on $[\alpha_i, \alpha_{i+1})$ but $\mathscr{I}(f, \alpha_i) \neq \mathscr{I}(f, \alpha_{i+1})$. The log-canonical threshold appears as α_1. These ideas had appeared previously in [137, 139].

Generalizing Kollar's approach, each α_i is a root of $b_f(s)$, [90]. In [193, Thm. 4.4] a partial converse is shown for locally Euler-homogeneous divisors. Extending the idea of jumping numbers to the range $\alpha > 1$ one sees that α is a jumping number if and only if $\alpha + 1$ is a jumping number, but the connection to the Bernstein–Sato polynomial is lost in general. For example, if $f(x, y) = x^2 + y^3$ then jumping numbers are $\{5/6, 1\} + \mathbb{N}$ while $b_f(s) = (s + 5/6)(s + 1)(s + 7/6)$.

3.3. **Bernstein–Sato polynomial.** The roots of $b_f(s)$ relate to an astounding number of other invariants, see for example [125] for a survey. However, besides the functional equation there is no known way to describe ρ_f.

3.3.1. *Fundamental results.* Let $p \in \mathbb{C}^n$ be a closed point, cut out by the maximal ideal $\mathfrak{m} \subseteq R$. Extending R to the localization $R_\mathfrak{m}$ (or even the ring of holomorphic functions at p) one arrives at potentially larger sets of polynomials $b_{f,P}(s)$ that satisfy a functional equation (1.2) with $P(x, \partial, s)$ now in the correspondingly larger ring of differential operators. The *local* (resp. *local analytic*) Bernstein–Sato polynomial $b_{f,p}(s)$ (resp. $b_{f,p^{an}}(s)$) is the generator of the resulting ideal generated by the $b_{f,P}(s)$ in $\mathbb{C}[s]$. We denote by $\rho_{f,p}$ (resp. $\tilde{\rho}_{f,p}$) the root set of $b_{f,p^{an}}(s)$ (resp. $b_{f,p^{an}}(s)/(s + 1)$). From the definitions and [143, 38, 36]

$$b_{f,p^{an}}(s) \big| b_{f,p}(s) \big| b_f(s) = \mathrm{lcm}_{p \in \mathrm{Var}(f)}\, b_{f,p}(s) = \mathrm{lcm}_{p \in \mathrm{Var}(f)}\, b_{f,p^{an}}(s), \tag{3.1}$$

and the function $\mathbb{C}^n \ni p \mapsto \mathrm{Var}(b_f(s))$, counting with multiplicity, is upper semi-continuous in the sense that for p' sufficiently near p one has $b_{f,p'}(s) | b_{f,p}(s)$. The underlying reason is the coherence of D.

The Bernstein–Sato polynomial $b_f(s)$ factors over $\mathbb{Q}$ into linear factors, $\rho_f \subseteq \mathbb{Q}$, and all roots are negative [146, 117]. The idea is to use resolution of singularities over $\mathbb{C}$ in order to reduce to simple normal crossing divisors, where rationality and negativity of the roots is evident. For this Kashiwara proves a comparison theorem [117, Thm. 5.1] that establishes $b_f(s)$ as a divisor of a shifted product of the least common multiple of the local Bernstein–Sato polynomials of the pullback of f under the resolution map. There is a refinement by Lichtin [135] for plane curves. The roots of $b_f(s)$, besides being negative, are always greater than $-n$, n being the minimum number of variables required to express f locally analytically [221, 191].

3.3.2. *Constructible sheaves from f^s.* Let $V = V(n, d)$ be the vector space of all complex polynomials in $x_1 \ldots, x_n$ of degree at most d. Consider the function $\beta \colon V \ni f \mapsto b_f(s)$. By [143, 36], there is an algebraic stratification of V such that on each stratum the function β is constant. For varying n, d these stratifications can be made to be compatible.

3.3.3. *Special cases.* If p is a smooth point of $\mathrm{Var}(f)$ then f can be used as an analytic coordinate near p, hence $b_{f,p^{an}}(s) = s+1$, and so $b_f(s) = s+1$ for all smooth hypersurfaces. By Proposition 2.6 in [35], an extension of [37], the equation $b_f(s) = s+1$ implies smoothness of $\mathrm{Var}(f)$. Explicit formulæ for the Bernstein–Sato polynomial are rare; here are some classes of examples.

- $f = \prod x_i^{a_i}$: $P = \prod \partial_i^{a_i}$ up to a scalar, $b_f(s) = \prod_i \prod_{j=1}^{a_i}(s + j/a_j)$.
- f (quasi-)homogeneous with isolated singularity at zero: $\tilde{b}_f(s) = \mathrm{lcm}(s + \frac{\deg(g dx)}{\deg(f)}$, where g runs through a (quasi-)homogeneous standard basis for J_f by work of Kashiwara, Sato, Miwa, Malgrange, Kochman [146, 235, 215, 124]. Note that the Jacobian ring of such a singularity is an Artinian Gorenstein ring, whose duality operator implies symmetry of ρ_f.
- $f = \det(x_{i,j})_1^n$: $P = \det(\partial_{i,j})_1^n$, $b_f(s) = (s+1)\cdots(s+n)$. This is attributed to Cayley, but see the comments in [63].
- For some hyperplane arrangements, $b_f(s)$ is known, see [230, 56].
- There is a huge list of examples worked out in [235].

If V is a complex vector space, G a reductive group acting linearly on V with open orbit U such that $V \smallsetminus U$ is a divisor $\mathrm{Var}(f)$, Sato's theory of prehomogeneous vectors spaces [198, 156, 197, 234] yields a factorization for $b_f(s)$. For reductive linear free divisors, [97, 203] discuss symmetry properties of Bernstein–Sato polynomials. In [162] this theme is taken up again, investigating specifically symmetry properties of ρ_f when $D[s] \bullet f^s$ has a Spencer logarithmic resolution (see [66] for definitions). This covers locally quasi-homogeneous free divisors, and more generally free divisors whose Jacobian is of linear type. The motivation is the fact that roots of $b_f(s)$ seem to come in strands, and whenever roots can be understood the strands appear to be linked to Hodge-theory.

There are several results on ρ_f for other divisors of special shape. Trivially, if $f(x) = g(x_1, \ldots, x_k) \cdot h(x_{k+1}, \ldots, x_n)$ then $b_f(s) \mid b_g(s) \cdot b_h(s)$; the question of equality appears to be open. In contrast, $b_f(s)$ cannot be assembled from the Bernstein–Sato polynomials of the factors of f in general, even if the factors are hyperplanes and one has some control on the intersection behavior, see Section 8 below. If $f(x) = g(x_1, \ldots, x_k) + h(x_{k+1}, \ldots, x_n)$ and at least one is locally Euler-homogeneous then there are Thom–Sebastiani type formulæ [191]. In particular, diagonal hypersurfaces are completely understood.

3.3.4. *Relation to intersection homology module.* Suppose $Y = \mathrm{Var}(f_1, \ldots, f_k) \subseteq X$ is a complete intersection and denote by $\mathscr{H}_Y^k(\mathscr{O}_X)$ the unique (algebraic) local cohomology module of $\mathscr{O}_X$ along Y. Brylinski–Kashiwara [42, 43] defined $\mathscr{L}(Y, X) \subseteq \mathscr{H}_Y^k(\mathscr{O}_X)$, the *intersection homology $\mathscr{D}_X$-module of Y*, the smallest $\mathscr{D}_X$-module equal to $\mathscr{H}_Y^k(\mathscr{O}_X)$ in the generic point. See also [19]. The module $\mathscr{L}(X, Y)$ contains the fundamental class of Y in X [20].

Question 3.2. *When is $\mathscr{L}(X, Y) = \mathscr{H}_Y^k(\mathscr{O}_X)$?*

Equality is equivalent to $\mathscr{H}_Y^k(\mathscr{O}_X)$ being generated by the cosets of $\Delta/\prod_{i=1}^k f_i$ over $\mathscr{D}_X$ where Δ is the ideal generated by the k-minors of the Jacobian matrix of $f_1, \ldots, f_k$. A necessary condition is that $1/\prod_{i=1}^k f_i$ generates $\mathscr{H}_Y^k(\mathscr{O}_X)$, but this is not sufficient: consider $xy(x+y)(x+yz)$, where $\rho_f = -\{1/2, 3/4, 1, 1, 1, 5/4\}$. Indeed, by [217], equality can be characterized in terms of functional equations, as the following are equivalent at $p \in X$:

(1) $\mathscr{L}(X,Y) = \mathscr{H}^k_Y(\mathscr{O}_X)$ in the stalk;
(2) $\tilde{\rho}_{f,p} \cap \mathbb{Z} = \emptyset$;
(3) 1 is not an eigenvalue of the monodromy operator on the reduced cohomology of the Milnor fibers near p.

If $1/\prod_{i=1}^k f_i$ generates $R[1/\prod f_i]$ and $1/\prod_{i=1}^k f_i \in \mathscr{L}(X,Y)$ then $\tilde{b}_f(-1) \neq 0$, [217]. It seems unknown whether (irrespective of $1/\prod_{i=1}^k f_i$ generating $R[1/\prod f_i]$) the condition $\tilde{b}_f(-1) \neq 0$ is equivalent to $1/\prod_{i=1}^k f_i$ being in $\mathscr{L}(X,Y)$. See also [149] for a topological viewpoint (by the Riemann–Hilbert correspondence of Kashiwara and Mebkhout [119, 150], $\mathscr{L}(X,Y)$ corresponds to the intersection cohomology complex of Y on X [42] and $\mathscr{H}^k_Y(\mathscr{O}_X)$ to $\mathbb{C}_Y[n-k]$, [102, 117, 152]; equality then says: the link is a rational homology sphere). In [21], Barlet characterizes property (3) above in terms of currents for complexified real f. Equivalence of (1) and (3) for isolated singularities can be derived from [155, 39]; the general case can be shown using [189, 4.5.8] and the formalism of weights. For the case $k=1$, (1) requires irreducibility; in the general case, there is a criterion in terms of b-functions [217, 1.6, 1.10].

4. LCT and logarithmic ideal

4.1. **Logarithmic forms.** Let $X = \mathbb{C}^n$ be the analytic manifold, f a holomorphic function on X, and $Y = \operatorname{Var}(f)$ a divisor in X with $j\colon U = X \smallsetminus Y \hookrightarrow X$ the embedding. Let $\Omega^\bullet_X(*Y)$ denote the complex of differential forms on X that are (at worst) meromorphic along Y. By [102], $\Omega^\bullet_X(*Y) \to \mathbb{R}j_*\mathbb{C}_U$ is a quasi-isomorphism.

A form ω is *logarithmic* along Y if $f\omega$ and $f\mathrm{d}\omega$ are holomorphic; these ω form the logarithmic de Rham complex $\Omega^\bullet_X(\log Y)$ on X along Y. The complex $\Omega^\bullet_X(\log Y)$ was first used with great effect by Deligne [83] on normal crossing divisors in order to establish mixed Hodge structures, and later by Esnault and Viehweg in order to prove vanishing theorems [91]. A major reason for the success of normal crossings is that in that case $\Omega^i_X(\log Y)$ is a locally free module over $\mathscr{O}_X$. The logarithmic de Rham complex was introduced in [187].

4.2. **Free divisors.**

Definition 4.1. A divisor $\operatorname{Var}(f)$ is *free* if (locally) $\Omega^1_X(\log f)$ is a free $\mathscr{O}_X$-module.

For a non-smooth locally Euler-homogenous divisor, freeness is equivalent to the Jacobian ring $\mathscr{O}_X/J_f$ being a Cohen–Macaulay $\mathscr{O}_X$-module of codimension 2; in general, freeness is equivalent to the Tjurina algebra $R/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ being of projective dimension 2 or less over R. See [187, 2] for relations to determinantal equations. Free divisors have rather big singular locus, and are in some ways at the opposite end of the singularity zoo from isolated singularities. If $\Omega^1_X(\log f)$ is (locally) free, then $\Omega^i_X(\log f) \cong \bigwedge^i \Omega^i_X(\log f)$ and also (locally) free, [187]. A weakening is

Definition 4.2. A divisor $\operatorname{Var}(f)$ is *tame* if, for all $i \in \mathbb{N}$, (locally) $\Omega^i_X(\log f)$ has projective dimension at most i as a $\mathscr{O}_X$-module.

Plane curves are trivially free; surfaces in 3-space are trivially tame. Normal crossing divisors are easily shown to be free. Discriminants of (semi)versal deformations of an isolated complete intersection singularity (and some others) are free, [2, 3, 141, 188, 77, 44]. Unitary reflection arrangements are free [212].

Definition 4.3. The *logarithmic derivations* $\mathrm{Der}(-\log f)$ along $Y = \mathrm{Var}(f)$ are the $\mathbb{C}$-linear derivations $\theta \in \mathrm{Der}(\mathscr{O}_X;\mathbb{C})$ that satisfy $\theta \bullet f \in (f)$.

A derivation θ is logarithmic along Y if and only it is so along each component of the reduced divisor to Y [187]. The modules $\mathrm{Der}(-\log f)$ and $\Omega^1(\log f)$ are reflexive and mutually dual over R. Moreover, $\Omega^i(\log f)$ and $\Omega^{n-i}(\log f)$ are dual.

4.3. **LCT.**

Definition 4.4. If

$$\Omega^\bullet_X(\log Y) \to \Omega^\bullet_X(*Y) \tag{4.1}$$

is a quasi-isomorphism, we say that *LCT holds for* Y.

We recommend [161].

Remark 4.5. (1) This "Logarithmic Comparison Theorem", a property of a divisor, is very hard to check explicitly. No general algorithms are known, even in $\mathbb{C}^3$ (but see [74] for $n = 2$).

(2) LCT fails for rather simple divisors such as $f = x_1x_2 + x_3x_4$.

(3) If Y is a reduced normal crossing divisor, Deligne proved (4.1) to be a filtered (by pole filtration) quasi-isomorphism [82]; this provided a crucial step in the development of the theory of mixed Hodge structures [83].

(4) Limiting the order of poles in forms needed to capture all cohomology of U started with the seminal article [99] and continues, see for example [81, 87, 113].

(5) The free case was studied for example in [72]. But even in this case, LCT is not understood.

(6) If f is quasi-homogeneous with an isolated singularity at the origin, then LCT for f is equivalent to a topological condition (the link of f at the origin being a rational homology sphere), as well as an arithmetic one on the Milnor algebra of f, [104]. In [202], using the Gauß–Manin connection, this is extended to a list of conditions on an isolated hypersurface singularity, each one of which forces the implication [D has LCT] $\Rightarrow$ [D is quasi-homogeneous].

(7) For a version regarding more general connections, see [58].

A plane curve satisfies LCT if and only it is locally quasi-homogeneous, [61]. By [72], free locally quasi-homogeneous divisors satisfy LCT in any dimension. By [95], in dimension three, free divisors with LCT must be locally Euler-homogeneous. Conjecturally, LCT implies local Euler-homogeneity [61]. The converse is false, see for example [69]. The classical example of rotating lines with varying cross-ratio $f = xy(x + y)(x + yz)$ is free, satisfies LCT and is locally Euler-homogeneous, but only weakly quasi-homogeneous, [61]. In [73], the effect of the Spencer property on LCT is discussed in the presence of homogeneity conditions. For locally quasi-homogeneous divisors (or if the non-free locus is zero-dimensional), LCT implies (B_1), [66, 216]. In particular, LCT implies (B_1) for divisors with isolated singularities. In [96] quasi-homogeneity of isolated singularities is characterized in terms of a map of local cohomology modules of logarithmic differentials.

A free divisor is *linear free* if the (free) module $\operatorname{Der}(-\log f)$ has a basis of linear vector fields. In [93], linear free divisors in dimension at most 4 are classified, and for these divisors LCT holds at least on global sections. In the process, it is shown that LCT is implied if the Lie algebra of linear logarithmic vector fields is reductive. The example of $n \times n$ invertible upper triangular matrices acting on symmetric matrices [93, Ex. 5.1] shows that LCT may hold without the reductivity assumption. Linear free divisors appear naturally, for example in quiver representations and in the theory of prehomogeneous vector spaces and castling transformations [45, 196, 94]. Linear freeness is related to unfoldings and Frobenius structures [79].

Denote by $\operatorname{Der}_0(-\log f)$ the derivations θ with $\theta \bullet f = 0$. In the presence of a global Euler-homogeneity E on Y there is a splitting $\operatorname{Der}(-\log f) \cong R \cdot E \oplus \operatorname{Der}_0(-\log f)$. Reading derivations as operators of order one, $\operatorname{Der}_0(-\log f) \subseteq \operatorname{ann}_D(f^s)$. We write S for $\operatorname{gr}_{(0,1)}(D)$; if y_i is the symbol of ∂_i then we have $S = R[y]$.

Definition 4.6. The inclusion $\operatorname{Der}_0(-\log f) \hookrightarrow \operatorname{ann}_D(f^s)$, via the order filtration, defines a subideal of $\operatorname{gr}_{(0,1)}(\operatorname{ann}_D(f^s)) \subseteq \operatorname{gr}_{(0,1)}(D) = S$ called the *logarithmic ideal* L_f of $\operatorname{Var}(f)$.

Note that the symbols of $\operatorname{Der}(-\log f)$ are in the ideal $R \cdot y$, which has height n.

Definition 4.7. If $\operatorname{Der}(-\log f)$ has a generating set (as an R-module) whose symbols form a regular sequence on S, then Y is called *Koszul free*.

As $\operatorname{Der}(-\log f)$ has rank n, a Koszul free divisor is indeed free. Divisors in the plane [187] and locally quasi-homogeneous free divisors [59, 57] are Koszul free. In the case of normal crossings, this has been used to make resolutions for $D[s] \bullet f^s$ and $D[s]/D[s](\operatorname{ann}_{D[s]} f^s, f)$, [101]. A way to distill invariants from resolutions of $D[s] \bullet f^s$ is given in [9]. The logarithmic module $\tilde{M}^{\log f} = D/D \cdot \operatorname{Der}(-\log f)$ has in the Spencer case (see [66, 62]) a natural free resolution of Koszul type.

For Koszul-free divisors, the ideal $D \cdot \operatorname{Der}(-\log f)$ is holonomic [60]. By [93, Thm. 7.4], in the presence of freeness, the Koszul property is equivalent to the local finiteness of Saito's logarithmic stratification. This yields an algorithmic way to certify (some) free divisors as not locally quasi-homogeneous, since free locally quasi-homogeneous divisors are Koszul free. Based on similar ideas, one may devise a test for strong local Euler-homogeneity [93, Lem. 7.5]. See [60] and [216, §2] for relations of Koszul freeness to perversity of the logarithmic de Rham complex.

Castro-Jiménez and Ucha established conditions for $Y = \operatorname{Var}(f)$ to have LCT in terms of D-modules [67, 66, 68] for certain free f. For example, LCT is equivalent to (A_1) for Spencer free divisors. Calderón-Moreno and Narváez-Macarro [62] proved that free divisors have LCT if and only if the natural morphism $\mathscr{D}_X \otimes^L_{V^0(\mathscr{D}_X)} \mathscr{O}_X(Y) \to \mathscr{O}_X(*Y)$ is a quasi-isomorphism, $\mathscr{O}_X(Y)$ being the meromorphic functions with simple pole along f. For Koszul free Y, one has at least $\mathscr{D}_X \otimes^L_{V^0(\mathscr{D}_X)} \mathscr{O}_X(Y) \cong \mathscr{D}_X \otimes_{V^0(\mathscr{D}_X)} \mathscr{O}_X(Y)$. A similar condition ensures that the logarithmic de Rham complex is perverse [60, 62]. The two results are related by duality between logarithmic connections on $\mathscr{D}_X$ and the V-filtration [66, 62, 75].

It is unknown how LCT is related to (A_1) in general, but for quasi-homogeneous polynomials with isolated singularities the two conditions are equivalent, [216].

4.4. **Logarithmic linearity.**

Definition 4.8. We say that $f \in R$ satisfies (L_s) if the characteristic ideal of $\operatorname{ann}_D(f^s)$ is generated by symbols of derivations.

Condition (L_s) holds for isolated singularities [235], locally quasi-homogeneous free divisors [59], and locally strongly Euler-homogeneous tame divisors [231]. Also, (L_s) plus (B_1) yields (A_1) for locally Euler-homogeneous f by [117], see [216].

The logarithmic ideal supplies an interesting link between $\Omega^\bullet(\log f)$ and $\operatorname{ann}_D(f^s)$ via approximation complexes: if f is strongly locally Euler-homogeneous and also tame then the complex $(\Omega^\bullet(\log f)[y], y\,\mathrm{d}x)$ is a resolution of the logarithmic ideal L_f, and S/L_f is a Cohen–Macaulay domain of dimension $n+1$; if f is in fact free, S/L_f is a complete intersection [161, 231].

Question 4.9. *For locally Euler-homogeneous divisors, is* $\operatorname{ann}_D(f^s)$ *related to the cohomology of* $(\Omega^\bullet(\log f)[y], y\,\mathrm{d}x)$*?*

5. Characteristic variety

For $f \in R$ let U_f be the open set defined by $\mathrm{d}f \neq 0 \neq f$. By [117], $\mathscr{M}_f(s)$ is coherent over D (by [23] there is a functional equation, so s has a minimal polynomial modulo f), and the restriction of $\operatorname{charV}(D[s] \bullet f^s)$ to U_f is

$$\overline{\left\{(\xi, s\frac{\mathrm{d}f(\xi)}{f(\xi)}) \mid \xi \in U_f, s \in \mathbb{C}\right\}}^{\text{Zariski}}, \tag{5.1}$$

an $(n+1)$-dimensional involutive subvariety of T^*U_f, [120]. Ginsburg [92] gives a formula for the characteristic cycle of $D[s] \bullet m f^s$ in terms of an intersection process for holonomic sections m.

In favorable cases, more can be said. By [59], if the divisor is reduced, free and locally quasi-homogeneous then $\operatorname{ann}_{D[s]}(f^s)$ is generated by derivations, both $\mathscr{M}_f(s)$ and $\mathscr{N}_f(s)$ have Koszul–Spencer type resolutions, and in particular the characteristic varieties are complete intersections. In the more general case where f is locally strongly Euler-homogeneous and tame, $\operatorname{ann}_D(f^s)$ is still generated by order one operators and the ideal of symbols of $\operatorname{ann}_D(f^s)$ (and hence the characteristic ideal of $\mathscr{M}_f(s)$ as well) is a Cohen–Macaulay prime ideal, [231]. Under these hypotheses, the characteristic ideal of $\mathscr{N}_f(s)$ is Cohen–Macaulay but not prime.

5.1. **Stratifications.** By [115], the resolution theorem of Hironaka can be used to show that there is a stratification of $\mathbb{C}^n$ such that for each holonomic D-module M, $\operatorname{charC}(M) = \bigsqcup_{\sigma\in\Sigma} \mu(M,\sigma) T^*_\sigma$ where T^*_σ is the closure of the conormal bundle of the smooth stratum σ in $\mathbb{C}^n$ and $\mu(M,\sigma) \in \mathbb{N}$.

For $D[s] \bullet f^s / D[s] \bullet f^{s+1}$ Kashiwara proved that if one considers a Whitney stratification S for f (for example the "canonical" stratification in [78]) then the characteristic variety of the D-module $\mathscr{N}_f(s)$ is the union of the conormal varieties of the strata $\sigma \in S$, [235].

If one slices a pair (X, D) of a smooth space and a divisor with a hyperplane, various invariants of the divisor will behave well provided that the hyperplane is not "special". A prime example are Bertini and Lefschetz theorems. For D-modules, Kashiwara defined the notion of *non-characteristic restriction*: the smooth hypersurface H is non-characteristic for the D-module M if it meets each component of the characteristic variety of M transversally (see [177] for an exposition). The condition assures that the inverse image functor attached to the embedding $H \hookrightarrow X$ has no higher derived functors for M. In [86] these ideas are used to show that the V-filtration, and hence the multiplier ideals as well as nearby and vanishing cycle sheaves, behave nicely under non-characteristic restriction.

5.2. **Deformations.** Varchenko proved, via establishing constancy of Hodge numbers, that in a μ-constant family of isolated singularities, the spectrum is constant [223]. In [86] it is shown that the formation of the spectrum along the divisor $Y \subseteq X$ commutes with the intersection with a hyperplane transversal to any stratum of a Whitney regular stratification of D. Moreover, they derived a weak generalization of Varchenko's constancy results for certain deformations of non-isolated singularities.

In contrast, the Bernstein–Sato polynomial may not be constant along μ-constant deformations. Suppose $f(x) + \lambda g(x)$ is a 1-parameter family of plane curves with isolated singularities at the origin. If the Milnor number $\dim_{\mathbb{C}}(R/J_{(f+\lambda g)})$ is constant in the family, the singularity germs in the family are topologically equivalent [218]; for discussion see [88, §2]. However, in such a family, $b_f(s)$ may vary, it is a differential invariant: $f + \lambda g = x^4 + y^5 + \lambda xy^4$ has constant Milnor number 20, but the general curve (not quasi-homogeneous in any coordinate system, as $\rho_{f+\lambda g}$ is not symmetric about -1, see Subsection 3.3 above) has $-\rho_{f+\lambda g} = \{1\} \cup \frac{1}{20}\{9, 11, 13, 14, 17, 18, 19, 21, 22, 23, 26, 27\}$ while the special curve has $-\rho_f = -\rho_{f+\lambda g} \cup \{-31/20\} \smallsetminus \{-11/20\}$. See [64] for details and similar examples based on Newton polytope considerations, and [205] for deformations of plane diagonal curves.

6. Milnor fiber and monodromy

6.1. **Milnor fibers.** Let $B(p, \varepsilon)$ denote the ε-ball around $p \in \operatorname{Var}(f) \subseteq \mathbb{C}^n$. Milnor [155] proved that the diffeomorphism type of the open real manifold

$$M_{p,t_0,\varepsilon} = B(p, \varepsilon) \cap \operatorname{Var}(f - t_0)$$

is independent of ε, t_0 as long as $0 < |t_0| \ll \varepsilon \ll 1$. For $0 < \tau \ll \varepsilon \ll 1$ denote by M_p the fiber of the bundle $B(p, \varepsilon) \cap \{q \in \mathbb{C}^n \mid 0 < |f(q)| < \tau\} \to f(q)$.

The direct image functor for D-modules to the projection $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$, $(x, t) \mapsto t$ turns the $D_{x,t}$-module $\mathscr{B}_f$ into the *Gauß-Manin system* $\mathscr{H}_f$. The D-module restriction of $H^k(\mathscr{H}_f)$ to $t = t_0$ is the k-th cohomology of the Milnor fibers along $\operatorname{Var}(f)$ for $0 < |t_0| < \tau$.

Fix a k-cycle $\sigma \in H_p(\operatorname{Var}(f - t_0))$ and choose $\eta \in H^k(\mathscr{H}_f)$. Deforming σ to a k-cycle over t using the Milnor fibration, one can evaluate $\int_{\sigma_t} \eta$. The Gauß–Manin system has Fuchsian singularities and these periods are in the Nilsson class [148]. For example, the classical Gauß hypergeometric function saw the light of day the first time as solution to a system of differential equations attached to the variation of the Hodge structure on an elliptic curve (expressed as integrals of the first and second kind) [41]. In [177] this point of view is taken to be the starting point. The techniques explained there form the foundation for many connections between f^s and singularity invariants attached to $\operatorname{Var}(f)$.

In [46], a bijection (for $0 < \alpha \le 1$) is established between a subset of the jumping numbers of f at $p \in \operatorname{Var}(f)$ and the support of the *Hodge spectrum* [207]

$$\operatorname{Sp}(f) = \sum_{\alpha \in \mathbb{Q}} n_\alpha(f) t^\alpha,$$

with $n_\alpha(f)$ determined by the size of the α-piece of Hodge component of the cohomology of the Milnor fiber of f at p. See also [190, 221], and [206] for a survey on Hodge invariants. We refer to [49, 194] for many more aspects of this part of the story.

6.2. **Monodromy.** The vector spaces $H^k(M_{p,t_0,\varepsilon},\mathbb{C})$ form a smooth vector bundle over a punctured disk $\mathbb{C}^*$. The linear transformation $\mu_{f,p,k}$ on $H^k(M_{p,t_0,\varepsilon},\mathbb{C})$ induced by $p\mapsto p\cdot\exp(2\pi i\lambda)$ is the k-th monodromy of f at p. Let $\chi_{f,p,k}(t)$ denote the characteristic polynomial of $\mu_{f,p,k}$, set

$$e_{f,p,k}=\{\gamma\in\mathbb{C}\mid\gamma\text{ is an eigenvalue of }\mu_{f,p,k}\}$$

and put $e_{f,p}=\bigcup e_{f,p,k}$.

For most (in a quantifiable sense) divisors f with given Newton diagram, a combinatorial recipe can be given that determines the alternating product $\prod(\chi_{f,p,k}(t))^{(-1)^k}$ [222], similarly to A'Campo's formula in terms of an embedded resolution [1].

6.3. **Degrees, eigenvalues, and Bernstein–Sato polynomial.** By [147, 114], the exponential function maps the root set of the local analytic Bernstein–Sato polynomial of f at p onto $e_{f,p}$. The set $\exp(-2\pi i\tilde{\rho}_{f,p})$ is the set of eigenvalues of the monodromy on the Grothendieck–Deligne vanishing cycle sheaf $\phi_f(\mathbb{C}_{X,p})$. This was shown in [191] by algebraic microlocalization.

If f is an isolated singularity, the Milnor fiber M_f is a bouquet of spheres, and $H^{n-1}(M_f,\mathbb{C})$ can be identified with the Jacobian ring R/J_f. Moreover, if f is quasi-homogeneous, then under this identification R/J_f is a $\mathbb{Q}[s]$-module, s acting via the Euler operator, and $\tilde{\rho}_f$ is in bijection with the degree set of the nonzero quasi-homogeneous elements in R/J_f. For non-isolated singularities, most of this breaks down, since R/J_f is not Artinian in that case. However, for homogeneous f, consider the *Jacobian module*

$$H^0_{\mathfrak{m}}(R/J_f)=\{g+J_f\mid\exists k\in\mathbb{N},\forall i,x_i^kg\in J_f\}$$

and the canonical $(n-1)$-form

$$\eta=\sum_i x_i\mathrm{d}x_1\wedge\cdots\wedge\widehat{\mathrm{d}x_i}\wedge\cdots\wedge\mathrm{d}x_n.$$

Every class in $H^{n-1}(M_f;\mathbb{C})$ is of the form $g\eta$ for suitable $g\in R$, [230]. Moreover, let $\overline{g}\neq 0$ by a homogeneous element in the Jacobian module $H^0_{\mathfrak{m}}(R/J_f)$ and suppose that its degree $\deg(g\eta)=\deg(g)+\sum_i\deg(x_i)$ is between d and $2d$. Then, by [230, 231], $g\eta$ represents a nonzero cohomology class in $H^{n-1}(M_f,\mathbb{C})$ and there is a filtration on $H^{n-1}(M_f,\mathbb{C})$ induced by integration of $\mathscr{B}_f$ along $\partial_1,\dots,\partial_n$, with the following property: if $g\in R$ is the smallest degree homogeneous polynomial such that $g\eta$ represents a chosen element of $H^{n-1}(M_f,\mathbb{C})$ then $b_f(-(\deg(g\eta))/\deg(f))=0$.

6.4. **Zeta functions.** The zeta function $Z_f(s)$ attached to a divisor $f\in R$ is the rational function

$$Z_f(s)=\sum_{I\subseteq S}\chi(E_I^*)\prod_{i\in I}\frac{1}{N_is+\nu_i}$$

where $\pi\colon(Y,\bigcup_I E_i)\to(\mathbb{C}^n,\mathrm{Var}(f))$ is an embedded resolution of singularities, and N_i (resp. ν_i-1) are the multiplicities of E_i in $\pi^*(f)$ (resp. in the Jacobian of π). By results of Denef and Loeser [84], $Z_f(s)$ is independent of the resolution.

Conjecture 6.1 (Topological Monodromy Conjecture)**.**

(SMC) Any pole of $Z_f(s)$ is a root of the Bernstein–Sato polynomial $b_f(s)$.

(MC) Any pole of $Z_f(s)$ yields under exponentiation an eigenvalue of the monodromy operator at some $p\in\mathrm{Var}(f)$.

The strong version (SMC) implies (MC) by [146, 114]. Each version allows a generalization to ideals.

(SMC), formulated by Igusa [110] and Denef–Loeser [84] holds for

- reduced curves by [138] with a discussion on the nature of the poles by Veys [226, 225, 227];
- certain Newton-nondegenerate divisors by [140];
- some hyperplane arrangements (see Section 8);
- monomial ideals in any dimension by [106].

Additionally, Conjecture (MC) holds for

- bivariate ideals by Van Proeyen and Veys [220];
- all hyperplane arrangements by [54, 56];
- some partial cases: [11, 127] some surfaces; [13] quasi-ordinary power series; [136, 140] in certain Newton non-degenerate cases; [109, 123] for invariants of prehomogeneous vector spaces; [126] for nondegenerate surfaces.

Strong evidence for (MC) for $n = 3$ is procured in [228]. The articles [180, 164] explore what (MC) could mean on a normal surface as ambient space and gives some results and counterexamples to naive generalizations. See also [85] and the introductions of [31, 32] for more details in survey format.

A root of $b_f(s)$, a monodromy eigenvalue, and a pole of $Z_f(s)$ may have multiplicity; can the monodromy conjecture be strengthened to include multiplicities? This version of (SMC) was proved for reduced bivariate f in [138]; in [153, 154] it is proved for certain nonreduced bivariate f, and for some trivariate ones.

A different variation, due to Veys, of the conjecture is the following. Vary the definition of $Z_f(s)$ to $Z_{f;g}(s) = \sum_{I\subseteq S} \chi(E_I^*) \prod_{i\in I} \frac{1}{N_i s+\nu_i'}$ where ν_i' is the multiplicity of E_i in the pullback along π of some differential form g. (The standard case is when g is the volume form). Two questions arise: (1) varying over a suitable set G of forms g, can one generate all roots of $b_f(s)$ as poles of the resulting zeta functions? And if so, can one (2) do this such that the pole sets of all zeta functions so constructed are always inside ρ_f, so that

$$\rho_f = \{\alpha \mid \exists g \in G, \lim_{s\to\alpha} Z_{f;g}(s) = \infty\} \quad ?$$

Némethi and Veys [163, 164] prove a weak version: if $n = 2$ then monodromy eigenvalues are exponentials of poles of zeta functions from differential forms.

The following is discussed in [30]. For some ideals with $n = 2$, (1) is false for the topological zeta function (even for divisors: consider $xy^5 + x^3y^2 + x^4y$). For monomial ideals with two generators in $n = 2$, (1) is correct; with more than two generators it can fail. Even in the former case, (2) can be false.

7. Multi-variate versions

If $f = (f_1, \ldots, f_r)$ defines a map $f\colon \mathbb{C}^n \to \mathbb{C}^r$, several b-functions can be defined:

(1) The univariate Bernstein–Sato polynomial $b_f(s)$ attached to the ideal $(f) \subseteq R$ from [51].

(2) The multi-variate Bernstein–Sato polynomials $b_{f,i}(s)$ of all $b(s) \in \mathbb{C}[s_1, \ldots, s_r]$ such that there is an equation $P(x, \partial, s) \bullet f_i f^s = b(s)f^s$ in multi-index notation.

(3) The multi-variate Bernstein–Sato ideal $B_{f,\mu}(s)$ for $\mu \in \mathbb{N}^r$ of all $b(s) \in \mathbb{C}[s_1, \ldots, s_r]$ such that there is an equation $P(x, \partial, s) \bullet f^{s+\mu} = b(s)f^s$ in multi-index notation. The most interesting case is $\mu = \mathbf{1} = (1, \ldots, 1)$.

(4) The multi-variate Bernstein–Sato ideal $B_{f,\Sigma}(s)$ of all $b(s) \in \mathbb{C}[s_1, \ldots, s_r]$ that multiply f^s into $\sum D[s]f_i f^s$ in multi-index notation.

The Bernstein–Sato polynomial in (1) above has been studied in the case of a monomial ideal in [52] and more generally from the point of view of the Newton polygon in [53]. While the roots for monomial ideals do not depend just on the Newton polygon, their residue classes modulo $\mathbb{Z}$ do.

Non-triviality of the quantities in (2)-(4) have been established in [184, 185, 183], but see also [17]. The ideals $B_{f,\mu}(s)$ and $B_{f,\Sigma}(s)$ do not have to be principal, [219, 18]. In [184, 103] it is shown that $B_{f,\mu}(s)$ contains a polynomial that factors into linear forms with non-negative rational coefficients and positive constant term. Bahloul and Oaku [18] show that these ideals are local in the sense of (3.1).

The following would generalize Kashiwara's result in the univariate case as well as the results of Sabbah and Gyoja above.

Conjecture 7.1 ([48])**.** *The Bernstein–Sato ideal $B_{f,\mu}(s)$ is is generated by products of linear forms $\sum \alpha_i s_i + a$ with α_i, a non-negative rational and $a > 0$.*

For $n = 2$, partial results by Cassou-Noguès and Libgober exist [65]. In [48] it is further conjectured that the Malgrange–Kashiwara result, exponentiating $\rho_{f,p}$ gives $e_{f,p}$, generalizes: monodromy in this case is defined in [224], and Sabbah's specialization functor ψ_f from [186] takes on the rôle of the nearby cycle functor, and conjecturally exponentiating the variety of $B_{f,p}(s)$ yields the uniform support (near p) of Sabbah's functor. The latter conjecture would imply Conjecture 7.1.

Similarly to the one-variable case, if $V(n, d, m)$ is the vector space of (ordered) m-tuples of polynomials in $x_1, \ldots, x_n$ of degree at most d, there is an algebraic stratification of $V(n, d, m)$ such that on each stratum the function $V \ni f = (f_1, \ldots, f_m) \mapsto b_f(s)$ is constant. Corresponding results for the Bernstein–Sato ideal $B_{f,\mathbf{1}}(s)$ hold by [38].

8. Hyperplane arrangements

A *hyperplane arrangement* is a divisor of the form

$$\mathscr{A} = \prod_{i \in I} \alpha_i$$

where each α_i is a polynomial of degree one. We denote $H_i = \mathrm{Var}(\alpha_i)$. Essentially all information we are interested in is of local nature, so we assume that each α_i is a form so that $\mathscr{A}$ is *central*. If there is a coordinate change in $\mathbb{C}^n$ such that $\mathscr{A}$ becomes the product of polynomials in disjoint sets of variables, the arrangement is *decomposable*, otherwise it is *indecomposable*.

A *flat* is any (set-theoretic) intersection $\bigcap_{i \in J} H_i$ where $J \subseteq I$. The *intersection lattice* $L(\mathscr{A})$ is the partially ordered set consisting of the collection of all flats, with order given by inclusion.

8.1. Numbers and parameters.

Hyperplane arrangements satisfy (B_1) everywhere [230]. Arrangements satisfy (A_1) everywhere if they decompose into a union of a generic and a hyperbolic arrangement [214], and if they are tame [231]. Terao conjectured that all hyperplane arrangements satisfy (A_1); some of them fail (A_s), [231].

Apart from recasting various of the previously encountered problems in the world of arrangements, a popular study is the following: choose a discrete invariant I of a divisor. Does the function $\mathscr{A} \mapsto I(\mathscr{A})$ factor through the map $\mathscr{A} \mapsto L(\mathscr{A})$? Randell showed that if two arrangements are connected by a one-parameter family of arrangements which have the same intersection lattice, the complements are diffeomorphic [178] and the isomorphism type of the Milnor fibration is constant [179]. Rybnikov [181, 12] showed on the other hand that there are arrangements (even in the projective plane) with equal lattice but different complement. In particular, not all isotopic arrangements can be linked by a smooth deformation.

8.2. **LCT and logarithmic ideal.** The most prominent positive result is by Brieskorn: the *Orlik–Solomon algebra* $\mathrm{OS}(\mathscr{A}) \subseteq \Omega^\bullet(\log \mathscr{A})$ generated by the forms $\mathrm{d}\alpha_i/\alpha_i$ is quasi-isomorphic to $\Omega^\bullet(*\mathscr{A})$, hence to the singular cohomology algebra of $U_\mathscr{A}$, [40]. The relation with combinatorics was given in [175, 176]. For a survey on the Orlik–Solomon algebra, see [237]. The best known open problem in this area is

Conjecture 8.1 ([211])**.** $\mathrm{OS}(\mathscr{A}) \to \Omega^\bullet(\log \mathscr{A})$ *is a quasi-isomorphism.*

While the general case remains open, Wiens and Yuzvinsky [232] proved it for tame arrangements, and also if $n \leq 4$. The techniques are based on [72].

8.3. **Milnor fibers.** There is a survey article by Suciu on complements, Milnor fibers, and cohomology jump loci [208], and [49] contains further information on the topic. It is not known whether $L(\mathscr{A})$ determines the Betti numbers (even less the Hodge numbers) of the Milnor fiber of an arrangement. The first Betti number of the Milnor fiber $M_\mathscr{A}$ at the origin is stable under intersection with a generic hyperplane (if $n > 2$). But it is unknown whether the first Betti number of an arrangement in 3-space is a function of the lattice alone. By [89], this is so for collections of up to 14 lines with up to 5-fold intersections in the projective plane. See also [134] for the origins of the approach. By [50], a lower combinatorial bound for the λ-eigenspace of $H^1(M_\mathscr{A})$ is given under favorable conditions on L. If L satisfies stronger conditions, the bound is shown to be exact. In any case, [50] gives an algebraic, although perhaps non-combinatorial, formula for the Hodge pieces in terms of multiplier ideals.

By [174], the Betti numbers of $M_\mathscr{A}$ are combinatorial if $\mathscr{A}$ is generic. See also [76].

8.4. **Multiplier ideals.** Mustaţă gave a formula for the multiplier ideals of arrangements, and used it to show that the log-canonical threshold is a function of $L(\mathscr{A})$. The formula is somewhat hard to use for showing that each jumping number is a lattice invariant; this problem was solved in [55]. Explicit formulas in low dimensional cases follow from the spectrum formulas given there and in [236]. Teitler [210] improved Mustaţă's formula for multiplier ideals to not necessarily reduced hyperplane arrangements [158].

8.5. **Bernstein–Sato polynomials.** By [230], $\rho_\mathscr{A} \cap \mathbb{Z} = \{-1\}$; by [192], $\rho_\mathscr{A} \subseteq (-2, 0)$. There are few classes of arrangements with explicit formulæ for their Bernstein–Sato polynomial:

- Boolean (a normal crossing arrangement, locally given by $x_1 \cdots x_k$);
- hyperbolic (essentially an arrangement in two variables);
- generic (central, and all intersections of n hyperplanes equal the origin).

The first case is trivial, the second is easy, the last is [230] with assistance from [193]. Some interesting computations are in [56], and [48] has a partial confirmation of the multi-variable Kashiwara–Malgrange theorem.

8.6. **Zeta functions.** Budur, Mustaţă and Teitler [54] show: (MC) holds for arrangements, and in order to prove (SMC), it suffices to show the following conjecture.

Conjecture 8.2. *For all indecomposable central arrangements with d planes in n-space, $b_{\mathscr{A}}(-n/d) = 0$.*

The idea is to use the resolution of singularities obtained by blowing up the dense edges from [200]. The corresponding computation of the zeta function is inspired from [107, 108]. The number $-n/d$ does not have to be the log-canonical threshold. By [54], Conjecture 8.2 holds in a number of cases, including reduced arrangements in dimension 3. By [231] it holds for tame arrangements.

Examples of Veys (in 4 variables) show that (SMC) may hold even if Conjecture 8.2 were false in general, since $-n/d$ is not always a pole of the zeta function [56]. However, in these examples, $-n/d$ is in fact a root of $b_f(s)$.

For arrangements, each monodromy eigenvalue can be captured by zeta functions in the sense of Némethi and Veys, see Subsection 6.4, but not necessarily all of $\rho_{\mathscr{A}}$ (Veys and Walther, unpublished).

9. Positive characteristic

Let here $\mathbb{F}$ denote a field of characteristic $p > 0$. The theory of D-modules is rather different in positive characteristic compared to their behavior over the complex numbers. There are several reasons for this:

(1) On the downside, the ring D_p of $\mathbb{F}$-linear differential operators on $R_p = \mathbb{F}[x_1, \ldots, x_n]$ is no longer finitely generated: as an $\mathbb{F}$-algebra it is generated by the elements $\partial^{(\alpha)}$, $\alpha \in \mathbb{N}^n$, which act via $\partial^{(\alpha)} \bullet (x^\beta) = \binom{\beta}{\alpha} x^{\beta-\alpha}$.
(2) As a trade-off, one has access to the Frobenius morphism $x_i \mapsto x_i^p$, as well as the Frobenius functor $F(M) = R' \otimes_R M$ where R' is the $R-R$-bimodule on which R acts via the identity on the left, and via the Frobenius on the right. Lyubeznik [142] created the category of F-finite F-modules and proved striking finiteness results. The category includes many interesting D_p-modules, and all F-modules are D_p-modules.
(3) Holonomicity provides certain difficulties, see [29].

A most surprising consequence of Lyubeznik's ideas is that in positive characteristic the property (B_1) is meaningless: it holds for every $f \in R_p$, [6]. The proof uses in significant ways the difference between D_p and the Weyl algebra. In particular, the theory of Bernstein–Sato polynomials is rather different in positive characteristic. In [159] a sequence of Bernstein–Sato polynomials is attached to a polynomial f assuming that the Frobenius morphism is finite on R (*e.g.*, if $\mathbb{F}$ is finite or algebraically closed); these polynomials are then linked to test ideals, the finite characteristic counterparts to multiplier ideals. In [28] variants of our modules $\mathscr{M}_f(\gamma)$ are introduced and [168] shows that simplicity of these modules detects the F-thresholds from [160]. These are cousins of the jumping numbers of multiplier ideals and related to the Bernstein–Sato polynomial via base-p-expansions; see also [233]. The Kashiwara–Brylinski intersection homology modules in positive characteristic was shown to exist in positive characteristic by Blickle in his thesis, [27].

10. Appendix: Computability (by A. Leykin)

Computations around f^s can be carried out by hand in special cases. Generally, the computations are enormous and computers are required (although not often sufficient). One of the earliest such approaches are in [34, 4], but at least implicitly Buchberger's algorithm in a Weyl algebra was discussed as early as [70]. An algorithmic approach to the isolated singularities case [144] preceded the general algorithms based on Gröbner bases in a non-commutative setting outlined below.

10.1. **Gröbner bases.** The *monomials* $x^\alpha\partial^\beta$ with $\alpha,\beta\in\mathbb{N}^n$ form a $\mathbb{C}$-basis of D; expressing $p\in D$ as linear combination of monomials leads to its *normal form.* The monomial orders on the commutative monoid $[x,\partial]$ for which for all $i\in[n]$ the leading monomial of $\partial_i x_i = x_i\partial_i + 1$ is $x_i\partial_i$, can be used to run Buchberger's algorithm in D. Modifications are needed in improvements that exploit commutativity, but the naïve Buchberger's algorithm works without any changes. See [112] for more general settings in polynomial rings of solvable type. Surprisingly, the worst case complexity of Gröbner bases computations in Weyl algebras is *not* worse than in the commutative polynomial case: it is doubly exponential in the number of indeterminates [14, 100].

10.2. **Characteristic variety.** A weight vector $(u,v)\in\mathbb{Z}^n\times\mathbb{Z}^n$ with $u+v\geq 0$ induces a filtration of D,

$$F_i = \mathbb{C}\cdot\{x^\alpha\partial^\beta \mid u\cdot\alpha+v\cdot\beta\leq i\},\quad i\in\mathbb{Z}.$$

The (u,v)-*Gröbner deformation* of a left ideal $I\subseteq D$ is

$$\operatorname{in}_{(u,v)}(I) = \mathbb{C}\cdot\{\operatorname{in}_{(u,v)}(P)\mid P\in I\}\subseteq \operatorname{gr}_{(u,v)} D,$$

the ideal of *initial forms* of elements of I with respect to the given weight in the associated graded algebra. It is possible to compute Gröbner deformations in the homogenized Weyl algebra

$$D^h = D\langle h\rangle/\langle \partial_i x_i - x_i\partial_i - h^2, x_i h - h x_i, \partial_i h - h\partial_i, \mid 1\leq i\leq n\rangle$$

see [71, 172]. Gröbner deformations are the main topic of [195].

10.3. **Annihilator.** Recall the construction appearing in the beginning of §6.1: $D_{x,t}$ acts on $D[s]f^s$; in particular, the operator $-\partial_t t$ acts as multiplication by s. It is this approach that lead Oaku to an algorithm for $\operatorname{ann}_{D[s]}(f^s)$, $\operatorname{ann}_D(f^s)$ and $b_f(s)$, [170]. We outline the ideas.

Malgrange observed that

$$\operatorname{ann}_{D[s]}(f^s) = \operatorname{ann}_{D_{x,t}}(f^s)\cap D[s], \tag{10.1}$$

$$\text{with}\quad \operatorname{ann}_{D_{x,t}}(f^s) = \langle t-f, \partial_1 + \tfrac{\partial f}{\partial x_1}\partial_t, \ldots, \partial_n + \tfrac{\partial f}{\partial x_n}\partial_t\rangle \subseteq D_{x,t}. \tag{10.2}$$

The former can be found from the latter by eliminating t and ∂_t from the ideal

$$\langle s + t\partial_t\rangle + \operatorname{ann}_{D_{x,t}}(f^s)\subseteq D_{x,t}\langle s\rangle; \tag{10.3}$$

of course $s=-\partial_t t$ does not commute with t,∂_t here.

Oaku's method for $\operatorname{ann}_{D[s]}(f^s)$ accomplished the elimination by augmenting two commuting indeterminates:

$$(10.4)\qquad \begin{aligned}&\operatorname{ann}_{D[s]}(f^s) = I'_f \cap D[s],\\ &\qquad I'_f = \langle t-uf, \partial_1 + u\tfrac{\partial f}{\partial x_1}\partial_t, \ldots, \partial_n + u\tfrac{\partial f}{\partial x_n}\partial_t, uv-1\rangle \subseteq D_{x,t}[u,v].\end{aligned}$$

Now outright eliminate u, v. Note that I'_f is quasi-homogeneous if the weights are $t, u \rightsquigarrow -1$ and $\partial_t, v \rightsquigarrow 1$, all other variables having weight zero. The homogeneity of the input and the relation $[\partial_t, t] = 1$ assures the termination of the computation. The operators of weight 0 in the output (with $-\partial_t t$ replaced by s) generate $I'_f \cap D[s]$.

A modification given in [33] and used, *e.g.*, in [219], reduces the number of algebra generators by one. Consider the subalgebra $D\langle s, \partial_t\rangle \subset D_{x,t}$; the relation $[s, \partial_t] = \partial_t$ shows that it is of solvable type. According to [33],

$$(10.5)\qquad \begin{aligned}&\operatorname{ann}_{D[s]}(f^s) = I''_f \cap D[s],\\ &\qquad I''_f = \langle s + f\partial_t, \partial_1 + \tfrac{\partial f}{\partial x_1}\partial_t, \ldots, \partial_n + \tfrac{\partial f}{\partial x_n}\partial_t\rangle \subset D\langle s, \partial_t\rangle.\end{aligned}$$

Note that $I''_f = \operatorname{ann}_{D_{x,t}}(f^s) \cap D\langle s, \partial_t\rangle$. The elimination step is done as in [170]; the decrease of variables usually improves performance. An algorithm to decide (A_1) for arrangements is given in [5].

10.4. **Algorithms for the Bernstein–Sato polynomial.** As the minimal polynomial of s on $\mathscr{N}_f(s)$, $b_f(s)$ can be obtained by means of linear algebra as a syzygy for the normal forms of powers of s modulo $\operatorname{ann}_{D[s]}(f^s) + D[s]\cdot f$ with respect to any fixed monomial order on $D[s]$. Most methods follow this path, starting with [170]. Variations appear in [229, 171, 173]; see also [195].

A slightly different approach is to compute $b_f(s)$ without recourse to $\operatorname{ann}_{D[s]}(f^s)$, via a Gröbner deformation of the ideal $I_f = \operatorname{ann}_{D_{x,t}}(f^s)$ in (10.2) with respect to the weight $(-w, w)$ with $w = (0^n, 1) \in \mathbb{N}^{n+1}$: $\langle b_f(s)\rangle = \operatorname{in}_{(-w,w)}(I_f) \cap \mathbb{Q}[-\partial_t t]$. Here again, computing the minimal polynomial using linear algebra tends to provide some savings in practice.

In [128] the authors give a method to check specific numbers for being in ρ_f. A method for $b_f(s)$ in the prehomogeneous vector space setup is in [157].

10.5. **Stratification from $b_f(s)$.** The Gröbner deformation $\operatorname{in}_{(-w,w)}(I_f)$ in §10.4 can be refined as follows, see [22, Thm. 2.2]. Let $b(x, s)$ be nonzero in the polynomial ring $\mathbb{C}[x, s]$. Then $b(x, s) \in (\operatorname{in}_{(-w,w)} I_f) \cap \mathbb{C}[x, s]$ if and only if there exists $P \in D[s]$ satisfying the functional equation $b(x, s)f^s = Pff^s$. From this one can design an algorithm not only for computing the *local* Bernstein–Sato polynomial $b_{f,p}(s)$ for $p \in \operatorname{Var}(f)$, but also the stratification of $\mathbb{C}^n$ according to local Bernstein–Sato polynomials; see [165, 22] for various approaches. Moreover, one can compute the stratifications from Subsection 3.3.2, see [131].

For the ideal case, [8] gives a method to compute an intersection of a left ideal of an associative algebra over a field with a subalgebra, generated by a single element. An application is a method for the computation of the Bernstein-Sato polynomial of an ideal. Another such was given by Bahloul in [15], and a version on general varieties in [16].

10.6. **Multiplier ideals.** Consider polynomials $f_1, \ldots, f_r \in \mathbb{C}[x]$, let f stand for $(f_1, \ldots, f_r)$, s for $s_1, \ldots, s_r$, and f^s for $\prod_{i=1}^r f_i^{s_i}$. In this subsection, let $D_{x,t} = \mathbb{C}\langle x, t, \partial_x, \partial_t\rangle$ be the $(n+r)$-th Weyl algebra.

Consider $D_{x,t}(s) \bullet f^s \subseteq R_{x,t}[f^{-1}, s]f^s$ and put

$$t_j \bullet h(x, s_1, \ldots, s_j, \ldots, s_r)f^s = h(x, s_1, \ldots, s_j + 1, \ldots, s_r)f_j f^s,$$
$$\partial_{t_j} \bullet h(x, s_1, \ldots, s_j, \ldots, s_r)f^s = -s_j h(x, s_1, \ldots, s_j - 1, \ldots, s_r)f_j^{-1} f^s,$$

for $h \in \mathbb{C}[x][f^{-1}, s]$, generalizing the univariate constructions.

The *generalized Bernstein–Sato polynomial* $b_{f,g}(\sigma)$ of f at $g \in \mathbb{C}[x]$ is the monic univariate polynomial b of the lowest degree for which there exist $P_k \in D_{x,t}$ such that

$$b(\sigma)gf^s = \sum_{k=1}^{r} P_k g f_k f^s, \quad \sigma = -\left(\sum_{i=1}^{r} \partial_{t_i} t_i\right). \tag{10.6}$$

An algorithm for $b_{f,g}(\sigma)$ is an essential ingredient for the algorithms in [204, 22] that compute the jumping numbers and corresponding multiplier ideals for $I = \langle f_1, \ldots, f_r \rangle$. That $b_{f,g}(\sigma)$ is related to multiplier ideals was worked out in [51].

There are algorithms for special cases: monomial ideals [105], hyperplane arrangements [158], and determinantal ideals [111]. A *Macaulay2* package *MultiplierIdeals* by Teitler collects all available (in *Macaulay2*) implementations. See also [47].

10.7. **Software.** Algorithms for computing Bernstein–Sato polynomials have been implemented in *kan/sm1* [209], *Risa/Asir* [167], *dmod_lib* library [130] for *Singular* [80], and the *D-modules* package [133] for *Macaulay2* [98]. The best source of information of these is documentation in the current versions of the corresponding software. A relatively recent comparison of the performance for several families of examples is given in [129].

The following are articles by developers discussing their implementations: [166, 165, 171, 7, 130, 132, 22].

References

[1] Norbert A'Campo. La fonction zêta d'une monodromie. *Comment. Math. Helv.*, 50:233–248, 1975. 6.2

[2] A. G. Aleksandrov. Euler-homogeneous singularities and logarithmic differential forms. *Ann. Global Anal. Geom.*, 4(2):225–242, 1986. 4.2, 4.2

[3] A. G. Aleksandrov. Nonisolated hypersurface singularities. In *Theory of singularities and its applications*, volume 1 of *Adv. Soviet Math.*, pages 211–246. Amer. Math. Soc., Providence, RI, 1990. 4.2

[4] A. G. Aleksandrov and V. L. Kistlerov. Computer method in calculating b-functions of nonisolated singularities. In *Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989)*, volume 131 of *Contemp. Math.*, pages 319–335, Providence, RI, 1992. Amer. Math. Soc. 10

[5] J. Àlvarez Montaner, F. J. Castro-Jiménez, and J. M. Ucha. Localization at hyperplane arrangements: combinatorics and $\mathcal{D}$-modules. *J. Algebra*, 316(2):662–679, 2007. 10.3

[6] Josep Alvarez-Montaner, Manuel Blickle, and Gennady Lyubeznik. Generators of D-modules in positive characteristic. *Math. Res. Lett.*, 12(4):459–473, 2005. 9

[7] Daniel Andres, Michael Brickenstein, Viktor Levandovskyy, Jorge Martín-Morales, and Hans Schönemann. Constructive D-module theory with SINGULAR. *Math. Comput. Sci.*, 4(2-3):359–383, 2010. 10.7

[8] Daniel Andres, Viktor Levandovskyy, and Jorge Martín Morales. Principal intersection and Bernstein-Sato polynomial of an affine variety. In *ISSAC 2009—Proceedings of the 2009 International Symposium on Symbolic and Algebraic Computation*, pages 231–238. ACM, New York, 2009. 10.5

[9] Rémi Arcadias. Minimal resolutions of geometric $\mathcal{D}$-modules. *J. Pure Appl. Algebra*, 214(8):1477–1496, 2010. 4.3

[10] V. I. Arnolʹd, S. M. Guseĭn-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. I*, volume 82 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985. The classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous and Mark Reynolds. 3.2

[11] E. Artal Bartolo, P. Cassou-Noguès, I. Luengo, and A. Melle Hernández. Monodromy conjecture for some surface singularities. *Ann. Sci. École Norm. Sup. (4)*, 35(4):605–640, 2002. 6.4

[12] Enrique Artal Bartolo, Jorge Carmona Ruber, José Ignacio Cogolludo Agustín, and Miguel Ángel Marco Buzunáriz. Invariants of combinatorial line arrangements and Rybnikov's example. In *Singularity theory and its applications*, volume 43 of *Adv. Stud. Pure Math.*, pages 1–34. Math. Soc. Japan, Tokyo, 2006. 8.1

[13] Enrique Artal Bartolo, Pierrette Cassou-Noguès, Ignacio Luengo, and Alejandro Melle Hernández. Quasi-ordinary power series and their zeta functions. *Mem. Amer. Math. Soc.*, 178(841):vi+85, 2005. 6.4

[14] Matthias Aschenbrenner and Anton Leykin. Degree bounds for Gröbner bases in algebras of solvable type. *J. Pure Appl. Algebra*, 213(8):1578–1605, 2009. 10.1

[15] Rouchdi Bahloul. Algorithm for computing Bernstein-Sato ideals associated with a polynomial mapping. *J. Symbolic Comput.*, 32(6):643–662, 2001. Effective methods in rings of differential operators. 10.5

[16] Rouchdi Bahloul. Global generic Bernstein-Sato polynomial on an irreducible affine scheme. *Proc. Japan Acad. Ser. A Math. Sci.*, 79(9):146–149, 2003. 10.5

[17] Rouchdi Bahloul. Démonstration constructive de l'existence de polynômes de Bernstein-Sato pour plusieurs fonctions analytiques. *Compos. Math.*, 141(1):175–191, 2005. 7

[18] Rouchdi Bahloul and Toshinori Oaku. Local Bernstein-Sato ideals: algorithm and examples. *J. Symbolic Comput.*, 45(1):46–59, 2010. 7

[19] D. Barlet and M. Kashiwara. Le réseau L^2 d'un système holonome régulier. *Invent. Math.*, 86(1):35–62, 1986. 3.3.4

[20] Daniel Barlet. Familles analytiques de cycles et classes fondamentales relatives. In *Functions of several complex variables, IV (Sem. François Norguet, 1977–1979) (French)*, volume 807 of *Lecture Notes in Math.*, pages 1–24. Springer, Berlin, 1980. 3.3.4

[21] Daniel Barlet. Multiple poles at negative integers for $\int_A f^\lambda \square$ in the case of an almost isolated singularity. *Publ. Res. Inst. Math. Sci.*, 35(4):571–584, 1999. 3.3.4

[22] Christine Berkesch and Anton Leykin. Algorithms for Bernstein-Sato polynomials and multiplier ideals. In *ISSAC 2010—Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation*, pages 99–106. ACM, New York, 2010. 10.5, 10.6, 10.7

[23] I. N. Bernšteĭn. Analytic continuation of generalized functions with respect to a parameter. *Funkcional. Anal. i Priložen.*, 6(4):26–40, 1972. 1, 1.5, 5

[24] Jospeh Bernstein. Lecture notes on the theory of D-modules. *unpublished, mimeographed notes.* 1, 1

[25] J.-E. Björk. *Rings of differential operators*, volume 21 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1979. 1, 1

[26] Jan-Erik Björk. *Analytic $\mathcal{D}$-modules and applications*, volume 247 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1993. 1, 2

[27] Manuel Blickle. The intersection homology D-module in finite characteristic. *Math. Ann.*, 328(3):425–450, 2004. 9

[28] Manuel Blickle, Mircea Mustaţă, and Karen E. Smith. F-thresholds of hypersurfaces. *Trans. Amer. Math. Soc.*, 361(12):6549–6565, 2009. 9

[29] Rikard Bögvad. An analogue of holonomic $\mathcal{D}$-modules on smooth varieties in positive characteristics. *Homology Homotopy Appl.*, 4(2, part 1):83–116, 2002. The Roos Festschrift volume, 1. 3

[30] Bart Bories. Zeta functions and Bernstein–Sato polynomials for ideals in dimension two. *Rev. Mat. Complut.*, 26(2):753–772, 2013. 6.4

[31] Bart Bories. Zeta functions, bernstein-sato polynomials, and the monodromy conjecture. *Ph.D. Thesis, Katholieke Universiteit Leuven*, 2013. 6.4

[32] Bart Bories and Wim Veys. Igusa's p-adic local zeta function and the monodromy conjecture for non-degenerated surface singularities. *arXiv*, 1306.6012, 2013. 6.4

[33] J. Briançon and Ph. Maisonobe. Remarques sur l'ideal de Bernstein associé à des polynomes. Preprint, 2002. 10.3

[34] J. Briançon, M. Granger, Ph. Maisonobe, and M. Miniconi. Algorithme de calcul du polynôme de Bernstein: cas non dégénéré. *Ann. Inst. Fourier (Grenoble)*, 39(3):553–610, 1989. 10

[35] J. Briançon and Ph. Maisonobe. Caractérisation géométrique de l'existence du polynôme de Bernstein relatif. In *Algebraic geometry and singularities (La Rábida, 1991)*, volume 134 of *Progr. Math.*, pages 215–236. Birkhäuser, Basel, 1996. 3.3.3

[36] J. Briançon and H. Maynadier. Équations fonctionnelles généralisées: transversalité et principalité de l'idéal de Bernstein-Sato. *J. Math. Kyoto Univ.*, 39(2):215–232, 1999. 3.3.1, 3.3.2

[37] Joël Briançon, Yves Laurent, and Philippe Maisonobe. Sur les modules différentiels holonomes réguliers, cohérents relativement à une projection. *C. R. Acad. Sci. Paris Sér. I Math.*, 313(5):285–288, 1991. 3.3.3

[38] Joël Briançon, Philippe Maisonobe, and Michel Merle. Constructibilité de l'idéal de Bernstein. In *Singularities—Sapporo 1998*, volume 29 of *Adv. Stud. Pure Math.*, pages 79–95. Kinokuniya, Tokyo, 2000. 3.3.1, 7

[39] Egbert Brieskorn. Die Monodromie der isolierten Singularitäten von Hyperflächen. *Manuscripta Math.*, 2:103–161, 1970. 3.3.4

[40] Egbert Brieskorn. Sur les groupes de tresses [d'après V. I. Arnol′d]. In *Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401*, pages 21–44. Lecture Notes in Math., Vol. 317. Springer, Berlin, 1973. 8.2

[41] Egbert Brieskorn and Horst Knörrer. *Ebene algebraische Kurven*. Birkhäuser Verlag, Basel, 1981. 6.1

[42] J.-L. Brylinski. Modules holonomes à singularités régulières et filtration de Hodge. II. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 75–117. Soc. Math. France, Paris, 1983. 3.3.4, 3.3.4

[43] J.-L. Brylinski. La classe fondamentale d'une variété algébrique engendre le $\mathcal{D}$-module qui calcule sa cohomologie d'intersection. *Astérisque*, (130):260–271, 1985. Differential systems and singularities (Luminy, 1983). 3.3.4

[44] Ragnar-Olaf Buchweitz, Wolfgang Ebeling, and Hans-Christian Graf von Bothmer. Low-dimensional singularities with free divisors as discriminants. *J. Algebraic Geom.*, 18(2):371–406, 2009. 4.2

[45] Ragnar-Olaf Buchweitz and David Mond. Linear free divisors and quiver representations. In *Singularities and computer algebra*, volume 324 of *London Math. Soc. Lecture Note Ser.*, pages 41–77. Cambridge Univ. Press, Cambridge, 2006. 4.3

[46] Nero Budur. On Hodge spectrum and multiplier ideals. *Math. Ann.*, 327(2):257–270, 2003. 6.1

[47] Nero Budur. On the V-filtration of $\mathcal{D}$-modules. In *Geometric methods in algebra and number theory*, volume 235 of *Progr. Math.*, pages 59–70. Birkhäuser Boston, Boston, MA, 2005. 3.1, 3.1.1, 10.6

[48] Nero Budur. Bernstein–Sato ideals and local systems. *arXiv*, 1209.3725, v. 4, 2012. 7.1, 7, 8.5

[49] Nero Budur. Singularity invariants related to Milnor fibers: survey. In *Zeta functions in algebra and geometry*, volume 566 of *Contemp. Math.*, pages 161–187. Amer. Math. Soc., Providence, RI, 2012. (document), 3.1, 6.1, 8.3

[50] Nero Budur, Alexandru Dimca, and Morihiko Saito. First Milnor cohomology of hyperplane arrangements. In *Topology of algebraic varieties and singularities*, volume 538 of *Contemp. Math.*, pages 279–292. Amer. Math. Soc., Providence, RI, 2011. 8.3

[51] Nero Budur, Mircea Mustaţă, and Morihiko Saito. Bernstein–Sato polynomials of arbitrary varieties. *Compos. Math.*, 142(3):779–797, 2006. 1, 10.6

[52] Nero Budur, Mircea Mustaţă, and Morihiko Saito. Combinatorial description of the roots of the Bernstein–Sato polynomials for monomial ideals. *Comm. Algebra*, 34(11):4103–4117, 2006. 7

[53] Nero Budur, Mircea Mustaţă, and Morihiko Saito. Roots of Bernstein–Sato polynomials for monomial ideals: a positive characteristic approach. *Math. Res. Lett.*, 13(1):125–142, 2006. 7

[54] Nero Budur, Mircea Mustaţă, and Zach Teitler. The monodromy conjecture for hyperplane arrangements. *Geom. Dedicata*, 153:131–137, 2011. 6.4, 8.6, 8.6

[55] Nero Budur and Morihiko Saito. Jumping coefficients and spectrum of a hyperplane arrangement. *Math. Ann.*, 347(3):545–579, 2010. 8.4

[56] Nero Budur, Morihiko Saito, and Sergey Yuzvinsky. On the local zeta functions and the b-functions of certain hyperplane arrangements. *J. Lond. Math. Soc. (2)*, 84(3):631–648, 2011. With an appendix by Willem Veys. 3.3.3, 6.4, 8.5, 8.6

[57] F. Calderón-Moreno and L. Narváez-Macarro. Locally quasi-homogeneous free divisors are Koszul free. *Tr. Mat. Inst. Steklova*, 238(Monodromiya v Zadachakh Algebr. Geom. i Differ. Uravn.):81–85, 2002. 4.3

[58] F. J. Calderón Moreno and L. Narváez Macarro. On the logarithmic comparison theorem for integrable logarithmic connections. *Proc. Lond. Math. Soc. (3)*, 98(3):585–606, 2009. 7

[59] Francisco Calderón-Moreno and Luis Narváez-Macarro. The module $\mathcal{D}f^s$ for locally quasi-homogeneous free divisors. *Compositio Math.*, 134(1):59–74, 2002. 2, 4.3, 4.4, 5

[60] Francisco J. Calderón-Moreno. Logarithmic differential operators and logarithmic de Rham complexes relative to a free divisor. *Ann. Sci. École Norm. Sup. (4)*, 32(5):701–714, 1999. 2, 4.3

[61] Francisco J. Calderón Moreno, David Mond, Luis Narváez Macarro, and Francisco J. Castro Jiménez. Logarithmic cohomology of the complement of a plane curve. *Comment. Math. Helv.*, 77(1):24–38, 2002. 2, 4.3

[62] Francisco Javier Calderón Moreno and Luis Narváez Macarro. Dualité et comparaison sur les complexes de de Rham logarithmiques par rapport aux diviseurs libres. *Ann. Inst. Fourier (Grenoble)*, 55(1):47–75, 2005. 4.3

[63] Sergio Caracciolo, Alan D. Sokal, and Andrea Sportiello. Algebraic/combinatorial proofs of Cayley-type identities for derivatives of determinants and Pfaffians. *Adv. in Appl. Math.*, 50(4):474–594, 2013. 3.3.3

[64] Pierrette Cassou-Noguès. Racines de polyômes de Bernstein. *Ann. Inst. Fourier (Grenoble)*, 36(4):1–30, 1986. 5.2

[65] Pierrette Cassou-Noguès and Anatoly Libgober. Multivariable Hodge theoretical invariants of germs of plane curves. *J. Knot Theory Ramifications*, 20(6):787–805, 2011. 7

[66] F. J. Castro-Jiménez and J. M. Ucha. Free divisors and duality for $\mathcal{D}$-modules. *Tr. Mat. Inst. Steklova*, 238(Monodromiya v Zadachakh Algebr. Geom. i Differ. Uravn.):97–105, 2002. 2, 3.3.3, 4.3, 4.3

[67] F. J. Castro-Jiménez and J. M. Ucha-Enríquez. Explicit comparison theorems for $\mathcal{D}$-modules. *J. Symbolic Comput.*, 32(6):677–685, 2001. Effective methods in rings of differential operators. 2, 4.3

[68] F. J. Castro-Jiménez and J. M. Ucha-Enríquez. Testing the logarithmic comparison theorem for free divisors. *Experiment. Math.*, 13(4):441–449, 2004. 4.3

[69] F. J. Castro-Jiménez and J. M. Ucha-Enríquez. Logarithmic comparison theorem and some Euler homogeneous free divisors. *Proc. Amer. Math. Soc.*, 133(5):1417–1422 (electronic), 2005. 4.3

[70] F.J. Castro-Jimenez. Théorème de division pour les opérateurs différentiels et calcul des multiplicités. *Thèse de 3e cycle*, Mathématiques:Paris VII, 1984. 10

[71] F.J. Castro-Jimenez and Luis Narváez-Macarro. Homogenizing differential operators. *Prepublicaciones de la Facultad de Matematicas de la Universidad de Sevilla*, 36, 1997. 10.2

[72] Francisco J. Castro-Jiménez, Luis Narváez-Macarro, and David Mond. Cohomology of the complement of a free divisor. *Trans. Amer. Math. Soc.*, 348(8):3037–3049, 1996. 5, 4.3, 8.2

[73] Francisco-Jesús Castro-Jiménez, Jesús Gago-Vargas, María-Isabel Hartillo-Hermoso, and José-María Ucha. A vanishing theorem for a class of logarithmic $\mathcal{D}$-modules. In *Proceedings of the XVIth Latin American Algebra Colloquium (Spanish)*, Bibl. Rev. Mat. Iberoamericana, pages 171–184. Rev. Mat. Iberoamericana, Madrid, 2007. 2, 4.3

[74] Francisco-Jesús Castro-Jiménez and Nobuki Takayama. The computation of the logarithmic cohomology for plane curves. *J. Algebra*, 322(11):3839–3851, 2009. 1

[75] Francisco Jesús Castro-Jiménez and José María Ucha-Enríquez. Quasi-free divisors and duality. *C. R. Math. Acad. Sci. Paris*, 338(6):461–466, 2004. 4.3

[76] Daniel C. Cohen and Alexander I. Suciu. On Milnor fibrations of arrangements. *J. London Math. Soc. (2)*, 51(1):105–119, 1995. 8.3

[77] James Damon. On the legacy of free divisors: discriminants and Morse-type singularities. *Amer. J. Math.*, 120(3):453–492, 1998. 4.2

[78] James Damon and David Mond. $\mathcal{A}$-codimension and the vanishing topology of discriminants. *Invent. Math.*, 106(2):217–242, 1991. 5.1

[79] Ignacio de Gregorio, David Mond, and Christian Sevenheck. Linear free divisors and Frobenius manifolds. *Compos. Math.*, 145(5):1305–1350, 2009. 4.3

[80] Greuel Decker, W., G.-M.; Pfister, and H. G.; Schönemann. SINGULAR 3-1-6 — A computer algebra system for polynomial computations. 2012. http://www.singular.uni-kl.de. 10.7

[81] P. Deligne and A. Dimca. Filtrations de Hodge et par l'ordre du pôle pour les hypersurfaces singulières. *Ann. Sci. École Norm. Sup. (4)*, 23(4):645–656, 1990. 4

[82] Pierre Deligne. *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin, 1970. 3

[83] Pierre Deligne. Théorie de Hodge. II/III. *Inst. Hautes Études Sci. Publ. Math.*, (40/44):5–57/5–77, 1971/1974. 4.1, 3

[84] J. Denef and F. Loeser. Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques. *J. Amer. Math. Soc.*, 5(4):705–720, 1992. 6.4, 6.4

[85] Jan Denef. Report on Igusa's local zeta function. *Astérisque*, (201-203):Exp. No. 741, 359–386 (1992), 1991. Séminaire Bourbaki, Vol. 1990/91. 6.4

[86] A. Dimca, Ph. Maisonobe, M. Saito, and T. Torrelli. Multiplier ideals, V-filtrations and transversal sections. *Math. Ann.*, 336(4):901–924, 2006. 5.1, 5.2

[87] Alexandru Dimca. On the de Rham cohomology of a hypersurface complement. *Amer. J. Math.*, 113(4):763–771, 1991. 4

[88] Alexandru Dimca. *Singularities and topology of hypersurfaces*. Universitext. Springer-Verlag, New York, 1992. 5.2

[89] Alexandru Dimca, Denis Ibadula, and Daniela Anca Macinic. Pencil type line arrangements of low degree: classification and monodromy. *arXiv*, 1305.5092, v. 2, 2013. 8.3

[90] Lawrence Ein, Robert Lazarsfeld, Karen E. Smith, and Dror Varolin. Jumping coefficients of multiplier ideals. *Duke Math. J.*, 123(3):469–506, 2004. 3.2

[91] Hélène Esnault and Eckart Viehweg. *Lectures on vanishing theorems*, volume 20 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1992. 4.1

[92] V. Ginsburg. Characteristic varieties and vanishing cycles. *Invent. Math.*, 84(2):327–402, 1986. 5

[93] Michel Granger, David Mond, Alicia Nieto-Reyes, and Mathias Schulze. Linear free divisors and the global logarithmic comparison theorem. *Ann. Inst. Fourier (Grenoble)*, 59(2):811–850, 2009. 4.3, 4.3

[94] Michel Granger, David Mond, and Mathias Schulze. Free divisors in prehomogeneous vector spaces. *Proc. Lond. Math. Soc. (3)*, 102(5):923–950, 2011. 4.3

[95] Michel Granger and Mathias Schulze. On the formal structure of logarithmic vector fields. *Compos. Math.*, 142(3):765–778, 2006. 4.3

[96] Michel Granger and Mathias Schulze. Quasihomogeneity of isolated hypersurface singularities and logarithmic cohomology. *Manuscripta Math.*, 121(4):411–416, 2006. 4.3

[97] Michel Granger and Mathias Schulze. On the symmetry of b-functions of linear free divisors. *Publ. Res. Inst. Math. Sci.*, 46(3):479–506, 2010. 3.3.3

[98] Daniel R. Grayson and Michael E. Stillman. Macaulay 2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/. 10.7

[99] Phillip A. Griffiths. On the periods of certain rational integrals. I, II. *Ann. of Math. (2) 90 (1969), 460-495; ibid. (2)*, 90:496–541, 1969. 4

[100] D. Yu. Grigoriev and A. L. Chistov. Complexity of the standard basis of a D-module. *Algebra i Analiz*, 20(5):41–82, 2008. 10.1

[101] Michel Gros and Luis Narváez-Macarro. Cohomologie évanescente p-adique: calculs locaux. *Rend. Sem. Mat. Univ. Padova*, 104:71–90, 2000. 4.3

[102] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966. 3.3.4, 4.1

[103] Akihiko Gyoja. Bernstein-Sato's polynomial for several analytic functions. *J. Math. Kyoto Univ.*, 33(2):399–411, 1993. 7

[104] Martin P. Holland and David Mond. Logarithmic differential forms and the cohomology of the complement of a divisor. *Math. Scand.*, 83(2):235–254, 1998. 6

[105] J. A. Howald. Multiplier ideals of monomial ideals. *Trans. Amer. Math. Soc.*, 353(7):2665–2671 (electronic), 2001. 10.6

[106] Jason Howald, Mircea Mustaţă, and Cornelia Yuen. On Igusa zeta functions of monomial ideals. *Proc. Amer. Math. Soc.*, 135(11):3425–3433 (electronic), 2007. 6.4

[107] Jun-ichi Igusa. Complex powers and asymptotic expansions. I. Functions of certain types. *J. Reine Angew. Math.*, 268/269:110–130, 1974. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, II. 8.6

[108] Jun-ichi Igusa. Complex powers and asymptotic expansions. II. Asymptotic expansions. *J. Reine Angew. Math.*, 278/279:307–321, 1975. 8.6

[109] Jun-ichi Igusa. Local zeta functions of certain prehomogeneous vector spaces. *Amer. J. Math.*, 114(2):251–296, 1992. 6.4

[110] Jun-ichi Igusa. *An introduction to the theory of local zeta functions*, volume 14 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2000. 6.4

[111] Amanda Ann Johnson. *Multiplier ideals of determinantal ideals*. ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–University of Michigan. 10.6

[112] A. Kandri-Rody and V. Weispfenning. Noncommutative Gröbner bases in algebras of solvable type. *J. Symbolic Comput.*, 9(1):1–26, 1990. 10.1

[113] Yakov Karpishpan. Pole order filtration on the cohomology of algebraic links. *Compositio Math.*, 78(2):213–226, 1991. 4

[114] M. Kashiwara. Vanishing cycle sheaves and holonomic systems of differential equations. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 134–142. Springer, Berlin, 1983. 3.1.1, 3.1.1, 6.3, 6.4

[115] M. Kashiwara and P. Schapira. Micro-hyperbolic systems. *Acta Math.*, 142(1-2):1–55, 1979. 5.1

[116] Masaki Kashiwara. On the maximally overdetermined system of linear differential equations. I. *Publ. Res. Inst. Math. Sci.*, 10:563–579, 1974/75. 1

[117] Masaki Kashiwara. B-functions and holonomic systems. Rationality of roots of B-functions. *Invent. Math.*, 38(1):33–53, 1976/77. 1.6, 2, 2, 3.1.1, 3.3.1, 3.3.4, 4.4, 5

[118] Masaki Kashiwara. On the holonomic systems of linear differential equations. II. *Invent. Math.*, 49(2):121–135, 1978. 3.1.1

[119] Masaki Kashiwara. The Riemann-Hilbert problem for holonomic systems. *Publ. Res. Inst. Math. Sci.*, 20(2):319–365, 1984. 3.3.4

[120] Masaki Kashiwara. *D-modules and microlocal calculus*, volume 217 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2003. Translated from the 2000 Japanese original by Mutsumi Saito, Iwanami Series in Modern Mathematics. 1, 1, 5

[121] Masaki Kashiwara and Takahiro Kawai. On holonomic systems of microdifferential equations. III. Systems with regular singularities. *Publ. Res. Inst. Math. Sci.*, 17(3):813–979, 1981. 1

[122] Masaki Kashiwara and Takahiro Kawai. On holonomic systems of microdifferential equations. III. Systems with regular singularities. *Publ. Res. Inst. Math. Sci.*, 17(3):813–979, 1981. 3.1.1

[123] Tatsuo Kimura, Fumihiro Satō, and Xiao-Wei Zhu. On the poles of p-adic complex powers and the b-functions of prehomogeneous vector spaces. *Amer. J. Math.*, 112(3):423–437, 1990. 6.4

[124] Fred Kochman. Bernstein polynomials and Milnor algebras. *Proc. Nat. Acad. Sci. U.S.A.*, 73(8):2546, 1976. 3.3.3

[125] János Kollár. Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 221–287. Amer. Math. Soc., Providence, RI, 1997. 3.2, 3.3

[126] Ann Lemahieu and Lise Van Proeyen. Monodromy conjecture for nondegenerate surface singularities. *Trans. Amer. Math. Soc.*, 363(9):4801–4829, 2011. 6.4

[127] Ann Lemahieu and Willem Veys. Zeta functions and monodromy for surfaces that are general for a toric idealistic cluster. *Int. Math. Res. Not. IMRN*, (1):Art. ID rnn122, 11–62, 2009. 6.4

[128] V. Levandovskyy and J. Martín-Morales. Algorithms for checking rational roots of b-functions and their applications. *J. Algebra*, 352:408–429, 2012. 10.4

[129] Viktor Levandovskyy and Jorge Martín Morales. Computational D-module theory with SINGULAR, comparison with other systems and two new algorithms. In *ISSAC 2008*, pages 173–180. ACM, New York, 2008. 10.7

[130] Viktor Levandovskyy and Jorge Martin Morales. Algorithms for algebraic D-modules. Available at http://www.singular.uni-kl.de. 10.7

[131] Anton Leykin. Constructibility of the set of polynomials with a fixed Bernstein-Sato polynomial: an algorithmic approach. *J. Symbolic Comput.*, 32(6):663–675, 2001. Effective methods in rings of differential operators. 10.5

[132] Anton Leykin. D-modules for Macaulay 2. In *Mathematical Software (Beijing, 2002)*, pages 169–179. World Sci. Publ., River Edge, 2002. 10.7

[133] Anton Leykin and Harrison Tsai. Software package "D-modules for Macaulay 2". Available at http://people.math.gatech.edu/∼aleykin3/Dmodules. 10.7

[134] Anatoly Libgober. On combinatorial invariance of the cohomology of the Milnor fiber of arrangements and the Catalan equation over function fields. In *Arrangements of hyperplanes—Sapporo 2009*, volume 62 of *Adv. Stud. Pure Math.*, pages 175–187. Math. Soc. Japan, Tokyo, 2012. 8.3

[135] B. Lichtin. Poles of $|f(z,w)|^{2s}$ and roots of the b-function. *Ark. Mat.*, 27(2):283–304, 1989. 3.2, 3.3.1

[136] Ben Lichtin and Diane Meuser. Poles of a local zeta function and Newton polygons. *Compositio Math.*, 55(3):313–332, 1985. 6.4

[137] Joseph Lipman. Equimultiplicity, reduction, and blowing up. In *Commutative algebra (Fairfax, Va., 1979)*, volume 68 of *Lecture Notes in Pure and Appl. Math.*, pages 111–147. Dekker, New York, 1982. 3.2

[138] F. Loeser. Fonctions d'Igusa p-adiques et polynômes de Bernstein. *Amer. J. Math.*, 110(1):1–21, 1988. 6.4

[139] F. Loeser and M. Vaquié. Le polynôme d'Alexander d'une courbe plane projective. *Topology*, 29(2):163–173, 1990. 3.2

[140] François Loeser. Fonctions d'Igusa p-adiques, polynômes de Bernstein, et polyèdres de Newton. *J. Reine Angew. Math.*, 412:75–96, 1990. 6.4

[141] E. J. N. Looijenga. *Isolated singular points on complete intersections*, volume 77 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1984. 4.2

[142] Gennady Lyubeznik. F-modules: applications to local cohomology and D-modules in characteristic $p > 0$. *J. Reine Angew. Math.*, 491:65–130, 1997. 2

[143] Gennady Lyubeznik. On Bernstein-Sato polynomials. *Proc. Amer. Math. Soc.*, 125(7):1941–1944, 1997. 3.3.1, 3.3.2

[144] Philippe Maisonobe. $\mathcal{D}$-modules: an overview towards effectivity. In *Computer algebra and differential equations (1992)*, volume 193 of *London Math. Soc. Lecture Note Ser.*, pages 21–55. Cambridge Univ. Press, Cambridge, 1994. 10

[145] Philippe Maisonobe and Zoghman Mebkhout. Le théorème de comparaison pour les cycles évanescents. In *Éléments de la théorie des systèmes différentiels géométriques*, volume 8 of *Sémin. Congr.*, pages 311–389. Soc. Math. France, Paris, 2004. 3.1

[146] B. Malgrange. Le polynôme de Bernstein d'une singularité isolée. In *Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974)*, pages 98–119. Lecture Notes in Math., Vol. 459. Springer, Berlin, 1975. 1, 3.3.1, 3.3.3, 6.4

[147] B. Malgrange. Polynômes de Bernstein-Sato et cohomologie évanescente. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 243–267. Soc. Math. France, Paris, 1983. 3.1.1, 3.1.1, 3.1.1, 6.3

[148] Bernard Malgrange. Intégrales asymptotiques et monodromie. *Ann. Sci. École Norm. Sup. (4)*, 7:405–430 (1975), 1974. 6.1

[149] David B. Massey. Intersection cohomology, monodromy and the Milnor fiber. *Internat. J. Math.*, 20(4):491–507, 2009. 3.3.4

[150] Z. Mebkhout. Une autre équivalence de catégories. *Compositio Math.*, 51(1):51–88, 1984. 3.3.4

[151] Z. Mebkhout and L. Narváez-Macarro. La théorie du polynôme de Bernstein-Sato pour les algèbres de Tate et de Dwork-Monsky-Washnitzer. *Ann. Sci. École Norm. Sup. (4)*, 24(2):227–256, 1991. 1

[152] Zoghman Mebkhout. Local cohomology of analytic spaces. In *Proceedings of the Oji Seminar on Algebraic Analysis and the RIMS Symposium on Algebraic Analysis (Kyoto Univ., Kyoto, 1976)*, volume 12 (supplement), pages 247–256, 1976/77. 3.3.4

[153] A. Melle-Hernández, T. Torrelli, and Willem Veys. On 'maximal' poles of zeta functions, roots of b-functions, and monodromy Jordan blocks. *J. Topol.*, 2(3):517–526, 2009. 6.4

[154] A. Melle-Hernández, T. Torrelli, and Willem Veys. Monodromy Jordan blocks, b-functions and poles of zeta functions for germs of plane curves. *J. Algebra*, 324(6):1364–1382, 2010. 6.4

[155] John Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J., 1968. 3.3.4, 6.1

[156] Masakazu Muro. A note on the holonomic system of invariant hyperfunctions on a certain prehomogeneous vector space. In *Algebraic analysis, Vol. II*, pages 493–503. Academic Press, Boston, MA, 1988. 3.3.3

[157] Masakazu Muro. An algorithm to compute the b_P-functions via Gröbner bases of invariant differential operators on prehomogeneous vector spaces. *Sūrikaisekikenkyūsho Kōkyūroku*, (1171):68–81, 2000. Algorithms for D-modules (Japanese) (Kyoto, 2000). 10.4

[158] Mircea Mustaţă. Multiplier ideals of hyperplane arrangements. *Trans. Amer. Math. Soc.*, 358(11):5015–5023 (electronic), 2006. 8.4, 10.6

[159] Mircea Mustaţă. Bernstein-Sato polynomials in positive characteristic. *J. Algebra*, 321(1):128–151, 2009. 9

[160] Mircea Mustaţă, Shunsuke Takagi, and Kei-ichi Watanabe. F-thresholds and Bernstein-Sato polynomials. In *European Congress of Mathematics*, pages 341–364. Eur. Math. Soc., Zürich, 2005. 9

[161] L. Narváez Macarro. Linearity conditions on the Jacobian ideal and logarithmic-meromorphic comparison for free divisors. In *Singularities I*, volume 474 of *Contemp. Math.*, pages 245–269. Amer. Math. Soc., Providence, RI, 2008. (document), 2, 4.3, 4.4

[162] Luis Narvaez-Macarro. A duality approach to the symmetry of Bernstein–Sato polynomials of free divisors. *arXiv*, 1201.3594, 2013. 3.3.3

[163] András Némethi and Willem Veys. Monodromy eigenvalues are induced by poles of zeta functions: the irreducible curve case. *Bull. Lond. Math. Soc.*, 42(2):312–322, 2010. 6.4

[164] András Némethi and Willem Veys. Generalized monodromy conjecture in dimension two. *Geom. Topol.*, 16(1):155–217, 2012. 6.4

[165] Kenta Nishiyama and Masayuki Noro. Stratification associated with local b-functions. *J. Symbolic Comput.*, 45(4):462–480, 2010. 10.5, 10.7

[166] Masayuki Noro. An efficient modular algorithm for computing the global b-function. In *Mathematical software (Beijing, 2002)*, pages 147–157. World Sci. Publ., River Edge, NJ, 2002. 10.7

[167] Masayuki Noro, Takeshi Shimoyama, and Taku Takeshima. Computer algebra system risa/asir. Available at http://www.math.kobe-u.ac.jp/Asir/index.html. 10.7

[168] Luis Nunẽz Betancourt and Felipe Pérez. F-jumping and F-jacobian ideals for hypersurfaces. *arXiv*, 1302.3327, 2013. 9

[169] Luis Núñez-Betancourt. On certain rings of differentiable type and finiteness properties of local cohomology. *J. Algebra*, 379:1–10, 2013. 1

[170] Toshinori Oaku. Algorithms for the b-function and D-modules associated with a polynomial. *J. Pure Appl. Algebra*, 117/118:495–518, 1997. Algorithms for algebra (Eindhoven, 1996). 10.3, 10.3, 10.4

[171] Toshinori Oaku and Nobuki Takayama. Algorithms for D-modules—restriction, tensor product, localization, and local cohomology groups. *J. Pure Appl. Algebra*, 156(2-3):267–308, 2001. 10.4, 10.7

[172] Toshinori Oaku and Nobuki Takayama. Minimal free resolutions of homogenized D-modules. *J. Symbolic Comput.*, 32(6):575–595, 2001. Effective methods in rings of differential operators. 10.2

[173] Toshinori Oaku, Nobuki Takayama, and Uli Walther. A localization algorithm for D-modules. *J. Symbolic Comput.*, 29(4-5):721–728, 2000. Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998). 10.4

[174] Peter Orlik and Richard Randell. The Milnor fiber of a generic arrangement. *Ark. Mat.*, 31(1):71–81, 1993. 8.3

[175] Peter Orlik and Louis Solomon. Combinatorics and topology of complements of hyperplanes. *Invent. Math.*, 56(2):167–189, 1980. 8.2

[176] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. 8.2

[177] Frédéric Pham. *Singularités des systèmes différentiels de Gauss-Manin*, volume 2 of *Progress in Mathematics*. Birkhäuser Boston, Mass., 1979. With contributions by Lo Kam Chan, Philippe Maisonobe and Jean-Étienne Rombaldi. 5.1, 6.1

[178] Richard Randell. Lattice-isotopic arrangements are topologically isomorphic. *Proc. Amer. Math. Soc.*, 107(2):555–559, 1989. 8.1

[179] Richard Randell. Milnor fibrations of lattice-isotopic arrangements. *Proc. Amer. Math. Soc.*, 125(10):3003–3009, 1997. 8.1

[180] B. Rodrigues. On the monodromy conjecture for curves on normal surfaces. *Math. Proc. Cambridge Philos. Soc.*, 136(2):313–324, 2004. 6.4

[181] G. L. Rybnikov. On the fundamental group of the complement of a complex hyperplane arrangement. *Funktsional. Anal. i Prilozhen.*, 45(2):71–85, 2011. 8.1

[182] C. Sabbah. D-modules et cycles évanescents (d'après B. Malgrange et M. Kashiwara). In *Géométrie algébrique et applications, III (La Rábida, 1984)*, volume 24 of *Travaux en Cours*, pages 53–98. Hermann, Paris, 1987. 3.1.1

[183] C. Sabbah. Polynômes de Bernstein-Sato à plusieurs variables. In *Séminaire sur les équations aux dérivées partielles 1986–1987*, pages Exp. No. XIX, 6. École Polytech., Palaiseau, 1987. 7

[184] C. Sabbah. Proximité évanescente. I. La structure polaire d'un $\mathcal{D}$-module. *Compositio Math.*, 62(3):283–328, 1987. 7

[185] C. Sabbah. Proximité évanescente. II. Équations fonctionnelles pour plusieurs fonctions analytiques. *Compositio Math.*, 64(2):213–241, 1987. 7

[186] Claude Sabbah. Modules d'Alexander et $\mathcal{D}$-modules. *Duke Math. J.*, 60(3):729–814, 1990. 7

[187] Kyoji Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980. 4.1, 4.2, 4.2, 4.3

[188] Kyoji Saito. Primitive forms for a universal unfolding of a function with an isolated critical point. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 28(3):775–792 (1982), 1981. 4.2

[189] Morihiko Saito. Mixed Hodge modules. *Publ. Res. Inst. Math. Sci.*, 26(2):221–333, 1990. 3.3.4

[190] Morihiko Saito. On b-function, spectrum and rational singularity. *Math. Ann.*, 295(1):51–74, 1993. 6.1

[191] Morihiko Saito. On microlocal b-function. *Bull. Soc. Math. France*, 122(2):163–184, 1994. 3.1, 3.1.1, 3.3.1, 3.3.3, 6.3

[192] Morihiko Saito. Bernstein–Sato polynomials of hyperplane arrangements. *arXiv*, 0602527, 2006. 8.5

[193] Morihiko Saito. Multiplier ideals, b-function, and spectrum of a hypersurface singularity. *Compos. Math.*, 143(4):1050–1068, 2007. (document), 3.2, 8.5

[194] Morihiko Saito. On b-function, spectrum and multiplier ideals. In *Algebraic analysis and around*, volume 54 of *Adv. Stud. Pure Math.*, pages 355–379. Math. Soc. Japan, Tokyo, 2009. 6.1

[195] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama. *Gröbner deformations of hypergeometric differential equations*, volume 6 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 2000. 10.2, 10.4

[196] M. Sato and T. Kimura. A classification of irreducible prehomogeneous vector spaces and their relative invariants. *Nagoya Math. J.*, 65:1–155, 1977. 4.3

[197] Mikio Sato. Theory of prehomogeneous vector spaces (algebraic part)—the English translation of Sato's lecture from Shintani's note. *Nagoya Math. J.*, 120:1–34, 1990. Notes by Takuro Shintani, Translated from the Japanese by Masakazu Muro. 3.3.3

[198] Mikio Sato and Takuro Shintani. On zeta functions associated with prehomogeneous vector spaces. *Ann. of Math. (2)*, 100:131–170, 1974. 3.3.3

[199] Pierre Schapira. *Microdifferential systems in the complex domain*, volume 269 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985. 1

[200] Vadim Schechtman, Hiroaki Terao, and Alexander Varchenko. Local systems over complements of hyperplanes and the Kac-Kazhdan conditions for singular vectors. *J. Pure Appl. Algebra*, 100(1-3):93–102, 1995. 8.6

[201] Mathias Schulze. A criterion for the logarithmic differential operators to be generated by vector fields. *Proc. Amer. Math. Soc.*, 135(11):3631–3640 (electronic), 2007. 2

[202] Mathias Schulze. Logarithmic comparison theorem versus Gauss-Manin system for isolated singularities. *Adv. Geom.*, 10(4):699–708, 2010. 6

[203] Christian Sevenheck. Bernstein polynomials and spectral numbers for linear free divisors. *Ann. Inst. Fourier (Grenoble)*, 61(1):379–400, 2011. 3.3.3

[204] Takafumi Shibuta. Algorithms for computing multiplier ideals. *J. Pure Appl. Algebra*, 215(12):2829–2842, 2011. 10.6

[205] Colin Stahlke. *Bernstein-Polynom und Tjurinazahl von μ-konstant-Deformationen der Singularitäten x^a+y^b*. Bonner Mathematische Schriften [Bonn Mathematical Publications], 305. Universität Bonn Mathematisches Institut, Bonn, 1998. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1997. 5.2

[206] J. Steenbrink. Mixed Hodge structures and singularities: a survey. In *Géométrie algébrique et applications, III (La Rábida, 1984)*, volume 24 of *Travaux en Cours*, pages 99–123. Hermann, Paris, 1987. 6.1

[207] J. H. M. Steenbrink. The spectrum of hypersurface singularities. *Astérisque*, (179-180):11, 163–184, 1989. Actes du Colloque de Théorie de Hodge (Luminy, 1987). 6.1

[208] Alexander I. Suciu. Hyperplane arrangements and milnor fibrations. *arXiv*, 1301.4851, 2013. (document), 8.3

[209] Nobuki Takayama. `kan/sm1`: a computer algebra system for algebraic analysis. www.math.sci.kobe-u.ac.jp/KAN/. 10.7

[210] Zach Teitler. A note on Mustaţă's computation of multiplier ideals of hyperplane arrangements. *Proc. Amer. Math. Soc.*, 136(5):1575–1579, 2008. 8.4

[211] Hiroaki Terao. Forms with logarithmic pole and the filtration by the order of the pole. In *Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977)*, pages 673–685, Tokyo, 1978. Kinokuniya Book Store. 8.1

[212] Hiroaki Terao. Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula. *Invent. Math.*, 63(1):159–179, 1981. 4.2

[213] Tristan Torrelli. Polynômes de Bernstein associés à une fonction sur une intersection complète à singularité isolée. *Ann. Inst. Fourier (Grenoble)*, 52(1):221–244, 2002. 2

[214] Tristan Torrelli. On meromorphic functions defined by a differential system of order 1. *Bull. Soc. Math. France*, 132(4):591–612, 2004. 2, 8.1

[215] Tristan Torrelli. Closed formulae for a b-function associated with a weighted homogeneous isolated complete intersection singularity. *J. Algebra*, 286(1):35–40, 2005. 3.3.3

[216] Tristan Torrelli. Logarithmic comparison theorem and $\mathcal{D}$-modules: an overview. In *Singularity theory*, pages 995–1009. World Sci. Publ., Hackensack, NJ, 2007. (document), 2, 4.3, 4.3, 4.4

[217] Tristan Torrelli. Intersection homology $\mathcal{D}$-module and Bernstein polynomials associated with a complete intersection. *Publ. Res. Inst. Math. Sci.*, 45(2):645–660, 2009. 2, 3.3.4, 3.3.4

[218] Lê Dũng Tráng and C. P. Ramanujam. The invariance of Milnor's number implies the invariance of the topological type. *Amer. J. Math.*, 98(1):67–78, 1976. 5.2

[219] J. M. Ucha and F. J. Castro-Jiménez. On the computation of Bernstein-Sato ideals. *J. Symbolic Comput.*, 37(5):629–639, 2004. 7, 10.3

[220] Lise Van Proeyen and Willem Veys. The monodromy conjecture for zeta functions associated to ideals in dimension two. *Ann. Inst. Fourier (Grenoble)*, 60(4):1347–1362, 2010. 6.4

[221] A. N. Varčenko. Asymptotic Hodge structure on vanishing cohomology. *Izv. Akad. Nauk SSSR Ser. Mat.*, 45(3):540–591, 688, 1981. 3.3.1, 6.1

[222] A. N. Varchenko. Zeta-function of monodromy and Newton's diagram. *Invent. Math.*, 37(3):253–262, 1976. 6.2

[223] A. N. Varchenko. The complex singularity index does not change along the stratum $\mu =$ const. *Funktsional. Anal. i Prilozhen.*, 16(1):1–12, 96, 1982. 5.2

[224] J.-L. Verdier. Spécialisation de faisceaux et monodromie modérée. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 332–364. Soc. Math. France, Paris, 1983. 7

[225] W. Veys. On the poles of Igusa's local zeta function for curves. *J. London Math. Soc. (2)*, 41(1):27–32, 1990. 6.4

[226] Willem Veys. Poles of Igusa's local zeta function and monodromy. *Bull. Soc. Math. France*, 121(4):545–598, 1993. 6.4

[227] Willem Veys. Determination of the poles of the topological zeta function for curves. *Manuscripta Math.*, 87(4):435–448, 1995. 6.4

[228] Willem Veys. Vanishing of principal value integrals on surfaces. *J. Reine Angew. Math.*, 598:139–158, 2006. 6.4

[229] Uli Walther. Algorithmic computation of local cohomology modules and the local cohomological dimension of algebraic varieties. *J. Pure Appl. Algebra*, 139(1-3):303–321, 1999. Effective methods in algebraic geometry (Saint-Malo, 1998). 10.4

[230] Uli Walther. Bernstein-Sato polynomial versus cohomology of the Milnor fiber for generic hyperplane arrangements. *Compos. Math.*, 141(1):121–145, 2005. 1, 2, 3.3.3, 6.3, 8.1, 8.5

[231] Uli Walther. Logarithmic ideals, Bernstein–Sato polynomials, and monodromy on homogeneous divisors. *in preparation*, 2013. 4.4, 5, 6.3, 8.1, 8.6

[232] Jonathan Wiens and Sergey Yuzvinsky. De Rham cohomology of logarithmic forms on arrangements of hyperplanes. *Trans. Amer. Math. Soc.*, 349(4):1653–1662, 1997. 8.2

[233] Emily E. Witt and Daniel Hernandez. in progress. 2013. 9

[234] Tamaki Yano. On the holonomic system of f^s and b-function. In *Proceedings of the Oji Seminar on Algebraic Analysis and the RIMS Symposium on Algebraic Analysis (Kyoto Univ., Kyoto, 1976)*, volume 12, pages 469–480, 1976/77 supplement. 3.3.3

[235] Tamaki Yano. On the theory of b-functions. *Publ. Res. Inst. Math. Sci.*, 14(1):111–202, 1978. 3.2, 3.3.3, 4.4, 5.1

[236] Youngho Yoon. Spectrum of hyperplane arrangements in four variables. *arXiv*, 1211.1689, 2012. 8.4

[237] S. Yuzvinskiĭ. Orlik-Solomon algebras in algebra and topology. *Uspekhi Mat. Nauk*, 56(2(338)):87–166, 2001. 8.2

U. Walther, Purdue University, Dept. of Mathematics, 150 N. University St., West Lafayette, IN 47907, USA

E-mail address: walther@math.purdue.edu

A. Leykin, School of Mathematics, Georgia Institute of Technology, 686 Cherry St., Atlanta, GA 30332, USA

E-mail address: leykin@math.gatech.edu