THE JACOBIAN MODULE, THE MILNOR FIBER, AND THE D-MODULE GENERATED BY f^s

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Abstract. For a germ f on a complex manifold X, we introduce a complex derived from the Liouville form acting on logarithmic differential forms, and give an exactness criterion. We use this Liouville complex to connect properties of the D-module generated by f^s to homological data of the Jacobian ideal; specifically we show that for a large class of germs the annihilator of f^s is generated by derivations. Through local cohomology, we connect the cohomology of the Milnor fiber to the Jacobian module via logarithmic differentials.

In particular, we consider (not necessarily reduced) hyperplane arrangements: we prove a conjecture of Terao on the annihilator of $1/f$; we confirm in many cases a corresponding conjecture on the annihilator of f^s but we disprove it in general; we show that the Bernstein–Sato polynomial of an arrangement is not determined by its intersection lattice; we prove that arrangements for which the annihilator of f^s is generated by derivations fulfill the Strong Monodromy Conjecture, and that this includes as very special cases all arrangements of Coxeter and of crystallographic type, and all multi-arrangements in dimension 3.

Contents

1. Introduction

Let X be a smooth analytic space or $\mathbb{C}$-scheme with structure sheaf $\mathscr{O}_X$ and sheaf of $\mathbb{C}$-linear differential operators $\mathscr{D}_X$ on $\mathscr{O}_X$. Throughout, $f \in \mathscr{O}_X$ will be a regular analytic non-constant function on X, not necessarily reduced, with divisor

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$\operatorname{Div}(f)$. To any such f one can attach several invariants that measure the singularity structure of the hypersurface $\operatorname{Var}(f)$. In this article, we are particularly interested in the following:

(1) The (parametric) annihilator $\operatorname{ann}_{\mathscr{D}_X[s]}(f^s)$: if s is a new variable, one associates to f the multi-valued function f^s on the locus $\{\mathfrak{x} \in X \mid f(\mathfrak{x}) \neq 0\}$. The free $\mathscr{O}_X[f^{-1}, s]$-module generated by the symbol f^s allows a left $\mathscr{D}_X[s]$-structure via the inherited $\mathscr{O}_X[s]$-structure and the rule

$$\delta \bullet (\frac{g}{f^k} f^s) \quad = \quad \delta \bullet (\frac{g}{f^k}) f^s + \frac{sg}{f^{k+1}} \cdot \delta \bullet (f) f^s \tag{1.1}$$

for each $g(x,s)$ in the stalk $\mathscr{O}_{X,\mathfrak{x}}[s]$ and each $\mathbb{C}$-linear derivation δ on $\mathscr{O}_{X,\mathfrak{x}}[s]$. The left ideals

$$\begin{aligned} \operatorname{ann}_{\mathscr{D}_X[s]}(f^s) &= \{P \in \mathscr{D}_X[s] \mid P \bullet f^s = 0\}, \\ \operatorname{ann}_{\mathscr{D}_X}(f^s) &= \mathscr{D}_X \cap \operatorname{ann}_{\mathscr{D}_X[s]}(f^s) \end{aligned}$$

contain important information on the singularity structure of f.

(2) The Bernstein–Sato polynomial $b_f(s)$: for $X = \mathbb{C}^n$, this is the (monic) generator of the $\mathbb{C}[s]$-ideal consisting of all polynomials $b_P(s)$ appearing in a functional equation of the form

$$(P \cdot f) \bullet f^s \quad = \quad b_P(s) \cdot f^s \tag{1.2}$$

where $P \in \mathscr{D}_X[s]$. A theorem of Bernstein asserts that $b_f(s)$ is nontrivial, so that the root set

$$\rho_f = \{\alpha \in \mathbb{C} \mid b_f(\alpha) = 0\}$$

is finite, [1]. For $\mathfrak{x} \in X$ there are local and analytic versions $b_{f,\mathfrak{x}}(s)$ with coefficients in the corresponding local or convergent power series rings, and $b_f(s) = \operatorname{lcm}_{\mathfrak{x}\in X} b_{f,\mathfrak{x}}(s)$, compare [38].

(3) The Milnor fiber $M_{f,\mathfrak{x}}$ at a point $\mathfrak{x} \in \operatorname{Var}(f)$: let $B(\mathfrak{x}, \varepsilon)$ denote the ε-ball around $\mathfrak{x} \in \operatorname{Var}(f)$. Milnor proved that the diffeomorphism type $M_{f,\mathfrak{x}}$ of the open real manifold $M_{\mathfrak{x},t,\varepsilon} = B(\mathfrak{x}, \varepsilon) \cap \operatorname{Var}(f - t)$ is independent of ε, t as long as $0 < |t| \ll \varepsilon \ll 1$. For $0 < \tau \ll \varepsilon \ll 1$ there is a fiber bundle

$$M_{f,\mathfrak{x}} \hookrightarrow B(\mathfrak{x}, \varepsilon) \cap \{\mathfrak{q} \in \mathbb{C}^n \mid 0 < |f(\mathfrak{q})| < \tau\} \longrightarrow B(0_{\mathbb{C}^1}, \tau) \smallsetminus \{0_{\mathbb{C}^1}\}.$$

(4) Monodromy at $\mathfrak{x} \in \operatorname{Var}(f)$: the above fibration induces for all $k \in \mathbb{N}$ smooth vector bundles $H^k(M_{\mathfrak{x},t,\varepsilon}, \mathbb{C})$ over the base of the fibration. The linear transformation $\mu_{f,\mathfrak{x}}$ induced on $H^\bullet(M_{f,\mathfrak{x}})$ by lifting the path $t \rightsquigarrow t \cdot \exp(2\pi\sqrt{-1}\lambda)$ for $\lambda \in [0,1]$ is the monodromy of f at $\mathfrak{x}$. We denote $e_{f,\mathfrak{x}} = \{\gamma \in \mathbb{C} \mid \gamma \text{ is an eigenvalue of } \mu_{f,\mathfrak{x}}\}$.

The following are classical results on these invariants:

(i) $\bigcup_{\mathfrak{x} \text{ near } \mathfrak{x}_0} e_{f,\mathfrak{x}} = \exp(2\pi\sqrt{-1}\rho_{f,\mathfrak{x}_0})$, [37, 34];
(ii) $\rho_{f,\mathfrak{x}} \subseteq \mathbb{Q} \cap (-n, 0)$, [35, 42];

Logarithmic forms and $\operatorname{ann}_{\mathscr{D}_X}(f^s)$. In this note we prove structural results for $\operatorname{ann}_{\mathscr{D}_X}(f^s)$ in the presence of suitable homogeneities of $f \in \mathscr{O}_X$. If there is a vector field E with $E \bullet (f) = f$ then f is *Euler-homogeneous*, cf. Definition 2.7 for details and strengthenings. Our main tool are the sheaves $\Omega^\bullet_X(\log f)$ of *logarithmic differential forms* along f. If Ω^1_X is the sheaf of $\mathbb{C}$-linear differentials on $X = \mathbb{C}^n$, and setting $\Omega^i_X = \bigwedge^i \Omega^1_X$, then $\Omega^i(\log f)$ is the (reflexive) sheaf of differential forms ω, meromorphic along $\mathrm{Var}(f)$, such that $f\omega$ and $f\mathrm{d}(\omega)$ are holomorphic:

$$\Omega^i_X(\log f) = \{\omega \in \frac{1}{f}\Omega^i_X \mid \mathrm{d}f \wedge \omega \in \Omega^{i+1}_X\} = \{\omega \in \frac{1}{f}\Omega^i_X \mid \mathrm{d}(\omega) \in \frac{1}{f}\Omega^{i+1}_X\}.$$

This construction does not depend on the choice of the coordinate system, and if $f = gu$ for a local unit u then $\Omega^i_X(\log f) = \Omega^i_X(\log g)$. Thus, given an effective divisor $Y \subseteq X$, X not necessarily $\mathbb{C}^n$, one can define a sheaf $\Omega^i_X(\log Y)$ locally as $\Omega^i_U(\log f)$ for any local defining equation f for Y on the open affine set U in X. For varieties over $\mathbb{C}$, the construction is compatible with the algebraic-analytic comparison map.

The logarithmic forms of order $n-1$ induce a decomposition of X into disjoint locally closed sets, see Definition 2.3. If this is a (locally finite) stratification, f is called *Saito-holonomic*, cf. Definition 2.5. In the same article [41] where K. Saito suggested holonomicity, he also introduced the notion of *freeness*, characterized by $\Omega^1_X(\log f)$ being a locally free $\mathscr{O}_X$-module. Free divisors have many nice properties and many distinguished classes of divisors are free; this includes Coxeter arrangements, discriminants in certain prehomogeneous vector spaces, and discriminants in the base of a versal deformation of isolated (complete intersection) singularities. Calderon-Moreno and Narvaez-Macarro [10, 39] studied certain free divisors from the differential point of view. Part of this note is a generalization and sharpening of their results to a significantly larger class of divisors, the central property being tameness, compare Definition 3.8. Tame divisors include all free divisors, and all divisors in dimension three or less.

Since exterior products of logarithmic differential forms are logarithmic again, $\Omega^\bullet_X(\log f)$ is a complex of $\mathscr{O}_X$-modules with differential $\frac{\mathrm{d}f}{f}\wedge$, and one can define $\mathscr{O}_X$-submodules

$$\Omega^i_X(\log_0 f) \quad := \quad \ker\left(\mathrm{d}f \wedge (-) \colon \Omega^i_X(\log f) \longrightarrow \Omega^{i+1}_X(\log f)\right) \tag{1.3}$$

which *do depend* on the specific choice of the defining equation f for $\mathrm{Div}(f)$.

Liouville complexes. Let $\pi \colon T^*X \longrightarrow X$ be the canonical projection from the total space of the cotangent bundle on X. The wedge product with the Liouville form $y\,\mathrm{d}x$ (see Subsection 3.1 below) defines a $\pi_*\pi^*(\mathscr{O}_X)$-morphism

$$y\,\mathrm{d}x \colon \pi_*\pi^*(\Omega^{i-1}_X(\log_0 f)) \longrightarrow \pi_*\pi^*(\Omega^i_X(\log_0 f))$$

for all i. In this note we construct from this morphism the *Liouville complex* $C^\bullet_f$ of f, a global version of a certain approximation complex from [31]; it is the main object of study in Section 3. The terminal cohomology group of the complex $C^\bullet_f$ of (reflexive) $\pi_*\pi^*(\mathscr{O}_X)$-modules is naturally identified with the quotient of $\pi_*\pi^*(\mathscr{O}_X)$ by the *Liouville ideal* of f, denoted $\mathscr{L}_f$. If $Y = \mathrm{Div}(f)$ is a strongly Euler-homogeneous (effective) divisor (Def. 2.7) then the Liouville ideal $\mathscr{L}_f$ is independent of the choice of f and only depends on Y, cf. Remark 2.8.

Our main result on the Liouville complex is the following theorem; see Section 3 for details.

Theorem 1.1. *If $f \in \mathscr{O}_X$ is tame, Saito-holonomic and strongly Euler-homogeneous (but not necessarily reduced) then the Liouville complex is a resolution (of reflexive modules) for the Liouville ideal $\mathscr{L}_f$, and $\mathscr{L}_f$ is a Cohen–Macaulay prime ideal of dimension $n+1$.*

One consequence of the above theorem is that under the stated hypotheses the symmetric algebra of the Jacobian ideal agrees with the Rees algebra of f, and both have a linear resolution, Corollary 3.23. In a different direction one obtains information on differential invariants of f as the Liouville ideal supplies a link between $\Omega^\bullet_X(\log f)$ and $\operatorname{ann}_{\mathscr{D}_X}(f^s)$:

Theorem 1.2. *Suppose $f \in \mathscr{O}_{X,\mathfrak{x}}$ is tame, Saito-holonomic, strongly Euler-homogeneous with strong Euler field $E_\mathfrak{x}$ at $\mathfrak{x} \in X$, but is not necessarily reduced. Then $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}}(f^s)$ is generated by the $\mathbb{C}$-linear derivations $\delta\colon \mathscr{O}_{X,\mathfrak{x}} \longrightarrow \mathscr{O}_{X,\mathfrak{x}}$ for which $\delta \bullet (f) = 0$. In consequence, $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[s]}(f^s) = \mathscr{D}_{X,\mathfrak{x}}[s] \cdot (\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}}(f^s), E_\mathfrak{x} - s)$ is generated by order one operators.*

Hyperplane arrangements. Let $X = \mathbb{C}^n$ and denote by D_n the ring of $\mathbb{C}$-linear algebraic differential operators on X (*i.e.*, the n-th complex Weyl algebra). A very interesting class of divisors are hyperplane arrangements $\mathscr{A}$, defined by a product $f_\mathscr{A}$ of linear polynomials. In generalization to the results in [46] which discuss differential operators with constant coefficients, Terao conjectured around 2002 that $\operatorname{ann}_{D_n}(1/f_\mathscr{A})$ be generated by differential operators of order one whenever $f_\mathscr{A}$ is reduced. Corresponding speculations have been made about $\operatorname{ann}_{D_n}(f^s_\mathscr{A})$ in [47, 49]. We use our techniques to prove that for tame arrangements $\operatorname{ann}_{D_n}(f^s_\mathscr{A})$ is indeed generated by derivations. In consequence, Terao's conjecture must hold in the tame case, but using an approach via local cohomology we actually prove it in Theorem 5.3 for all arrangements, irrespective of tameness or multiplicities. On the other hand, we provide in Example 5.7 an arrangement for which $\operatorname{ann}_{D_n}(f^s_\mathscr{A})$ is *not* generated by derivations (while, of course, $\operatorname{ann}_{D_n}(1/f_\mathscr{A})$ still *is* generated by operators of order one).

Aside from the fact that -1 is always a root of the Bernstein–Sato polynomial $b_f(s)$ of a non-constant polynomial f, very little is known about specific roots. Budur, Mustaţă and Teitler formulated the following idea:

Conjecture 1.3 (The n/d-conjecture, [6])**.** *If $f_\mathscr{A}$ defines a central reduced indecomposable arrangement of d hyperplanes in $\mathbb{C}^n$ then $-n/d$ is a root of the Bernstein–Sato polynomial of $f_\mathscr{A}$.*

For any central indecomposable arrangement $\mathscr{A}$ (tame or otherwise) we prove in Theorem 5.13 that all derivations that kill $f_\mathscr{A}$ (or $1/f_\mathscr{A}$, or $f^s_\mathscr{A}$) lie in the ideal $D_n \cdot (x_1, \ldots, x_n)$. This in turn has the following consequence.

Theorem 1.4. *Let $f_\mathscr{A}$ be a central, indecomposable, not necessarily reduced, arrangement of degree d in n variables for which $\operatorname{ann}_{D_n}(f^s_\mathscr{A})$ is generated by derivations; for example, $\mathscr{A}$ could be tame. Then, the n/d-conjecture holds for $f_\mathscr{A}$.*

The interest in the n/d-Conjecture comes from the Strong (topological) Monodromy Conjecture. To state that, denote by $Z_f(s)$ the topological zeta function $Z_f(s)$ attached to a divisor $\operatorname{Div}(f)$. This is the rational function

$$Z_f(s) \quad = \quad \sum_{I \subseteq S} \chi(E_I^*) \prod_{i \in I} \frac{1}{N_i s + \nu_i} \tag{1.4}$$

where $\pi\colon (Y, \bigcup_S E_i) \longrightarrow (\mathbb{C}^n, \operatorname{Var}(f))$ is an embedded resolution of singularities, and N_i (resp. $\nu_i - 1$) are the multiplicities of E_i in $\pi^*(f)$ (resp. in the Jacobian of π). By results of Denef and Loeser in [21], $Z_f(s)$ is independent of the resolution.

The Strong (topological) Monodromy Conjecture claims that any pole of $Z_f(s)$ is a root of the Bernstein–Sato polynomial $b_f(s)$ (see [4] for more information). The Strong Monodromy Conjecture is a variant of an old (and still wide open) conjecture of Igusa linking p-adic integrals to the root set ρ_f, cf. [20].

Theorem 1.4 then implies, by ideas of Budur, Mustaţă and Teitler from [6]:

Corollary 1.5. *The Strong Monodromy Conjecture holds for all tame arrangements (irrespective of centrality, indecomposability or reducedness).*

This result extends in a new direction some of the results in [7] as every arrangement in dimension three or less is tame. It proves the Strong Monodromy Conjecture for all arrangements of the following types: Coxeter arrangements; Ziegler's multi-reflection arrangements; arrangements of crystallographic type; all multi-arrangements in dimension 3.

A folklore conjecture states that the Bernstein–Sato polynomial of a hyperplane arrangement $\mathscr{A}$ depends only on the intersection lattice. Inspired by Theorem 1.6 discussed below we construct in Section 4 a pair of arrangements that defeats this conjecture. However, we offer an improved conjecture involving a slightly finer combinatorial invariant than the usual intersection lattice. We describe next our approach towards Theorem 1.6.

Jacobian module and monodromy. Let $X = \mathbb{C}^n$ and set $R_n = \mathbb{C}[x_1, \dots, x_n]$ with maximal homogeneous ideal $\mathfrak{m}$. If f has an isolated singularity at the origin, the Milnor fiber M_f at the origin is a bouquet of spheres, and in the Euler-homogeneous case the number of these spheres equals the $\mathbb{C}$-dimension of the Jacobian ring $R_n/\operatorname{Jac}(f)$. Malgrange showed that for Euler-homogeneous isolated singularities $R_n/\operatorname{Jac}(f)$ has a $\mathbb{Q}[s]$-module structure where s acts via the Euler-homogeneity, [37]. If f is, in addition, quasi-homogeneous then the root set of $b_f(s)/(s+1)$ is in bijection with the degree set of the nonzero quasi-homogeneous elements in $R_n/\operatorname{Jac}(f)$. For positive-dimensional singular loci, much of this breaks down, since $R_n/\operatorname{Jac}(f)$ is not Artinian in that case. Let the *Jacobian module* be

$$H^0_{\mathfrak{m}}(R_n/\operatorname{Jac}(f)) = \{g + \operatorname{Jac}(f) \mid \exists k \in \mathbb{N}, \forall i\ x_i^k g \in \operatorname{Jac}(f)\}.$$

We give in Section 4 the following generalization of Malgrange's result which has recently been used by A. Dimca and G. Sticlaru in [24].

Theorem 1.6. *Let $f \in R_n$ be reduced and homogeneous of degree d, with $n \geq 2$. Assume that $\operatorname{Proj}(R_n/f)$ has isolated singularities. Then, with $1 \leq k \leq d$ and $\lambda = \exp(2\pi\sqrt{-1}k/d)$,*

$$\dim_{\mathbb{C}}[H^0_{\mathfrak{m}}(R_n/\operatorname{Jac}(f))]_{d-n+k} \quad \leq \quad \dim_{\mathbb{C}} \operatorname{gr}^{\text{Hodge}}_{n-2}(H^{n-1}(M_f, \mathbb{C})_\lambda)$$

where the right hand side indicates the λ-eigenspace of the associated graded object to the Hodge filtration on $H^{n-1}(M_f, \mathbb{C})$, and where the left hand side denotes the graded component of the Jacobian module.

2. Stratifications and homogeneity

We collect in this section useful facts about stratifications and homogeneity properties. As always, X is a complex manifold with structure sheaf $\mathscr{O}_X$ of holomorphic functions. Points of X are usually denoted $\mathfrak{x}$ and the stalk of an object at $\mathfrak{x}$ by $(-)_{\mathfrak{x}}$.

2.1. Stratifications.

Whitney stratifications. Whitney [50] showed that all closed analytic spaces $Y \subseteq X$ have a *Whitney stratification*, satisfying certain conditions on limits of tangent spaces of strata. If $Y' \subseteq Y$ are closed analytic, there exist Whitney stratifications $\Sigma_{Y,Y'}$ of Y such that each stratum is either contained in or disjoint to Y', and in addition the strata inside Y' form a Whitney stratification for Y'.

Each analytic space, except for finite sets of points, has infinitely many Whitney stratifications. There is however, one distinguished such:

Definition 2.1. The *canonical Whitney stratification* Σ_Y of the analytic set Y is defined inductively as follows:

- W_0 is the regular part of Y;
- if Y^k denotes $Y \smallsetminus \bigcup_{i=0}^{k-1} W_i$ then W_k is for $k > 0$ the points of the regular part Y^k_{reg} of Y^k at which the Whitney conditions for the pairs $(W_i, Y^k_{\mathrm{reg}})$ are satisfied for all $i < k$.

Logarithmic stratification. If δ is an analytic vector field on X then integrating δ leads to a foliation of X into curves near any point where δ does not vanish. The following definition was made by K. Saito for divisors Y.

Definition 2.2. For an ideal sheaf $\mathscr{I}_Y \subseteq \mathscr{O}_X$, let $\mathrm{Der}_X(-\log Y)$ be the sheaf generated locally by the vector fields δ with $\delta \bullet (\mathscr{I}_Y) \subseteq \mathscr{I}_Y$.

It follows from the product rule that logarithmic derivations are indeed a function of the ideal and not of a set of chosen generators. If f cuts out Y we use $\mathrm{Der}_X(-\log Y)$ and $\mathrm{Der}_X(-\log f)$ interchangeably, but we distinguish between logarithmic data along Y and its reduced scheme Y_{red}.

For a reduced divisor $Y = Y_{\mathrm{red}}$, K. Saito [41] introduced logarithmic derivations, and also the following concept.

Definition 2.3. Pick $\mathfrak{x}, \mathfrak{x}' \in X$ and write $\mathfrak{x} \sim_Y \mathfrak{x}'$ if there exist an open set $U \subseteq X$ containing $\mathfrak{x}, \mathfrak{x}'$ and a derivation $\delta \in \mathrm{Der}_U(-\log(Y \cap U))$, vanishing nowhere on U, such that one of the integrating curves of δ passes through both $\mathfrak{x}$ and $\mathfrak{x}'$.

Varying over all open $U \subseteq X$ and all $\delta \in \mathrm{Der}_U(-\log(Y \cap U))$ one induces an equivalence relation denoted $\mathfrak{x} \approx_Y \mathfrak{x}'$. The strata of the *logarithmic stratification* $\mathfrak{S}_{X,Y}$ of X induced by Y are then by definition the irreducible components of the cosets of the relation $\approx_Y$. The set of strata includes $X \smallsetminus Y$, and the components of the non-singular locus of Y_{red}.

Saito noted cases where this stratification is not locally finite: the dimension of $\{\mathfrak{x} \in X \mid \mathrm{rk}_{\mathbb{C}}(\mathrm{Der}_X(-\log Y) \otimes \mathscr{O}_{X,\mathfrak{x}}/\mathfrak{m}_{\mathfrak{x}}) = i\}$ can be greater than i.

Example 2.4. Let $Y = \mathrm{Var}(xy(x+y)(x+zy)) \subseteq X = \mathbb{C}^3$. Then $\mathrm{Der}_X(-\log Y)$ vanishes identically on the z-axis and the z-axis is an irreducible component of $\{\mathfrak{x} \in X \mid \mathrm{rk}_{\mathbb{C}}(\mathrm{Der}_X(-\log Y) \otimes_{\mathscr{O}_X} \mathscr{O}_{X,\mathfrak{x}}/\mathfrak{m}_{\mathfrak{x}}) = 0\}$. Each point on the z-axis is its own logarithmic stratum. ⋄

Definition 2.5 ([41])**.** If the logarithmic stratification induced by Y on X is everywhere locally finite then Y is called *Saito-holonomic.*

From now on, Y will be a divisor, not necessarily reduced.

Remark 2.6. (1) Let σ be a positive-dimensional stratum in a Whitney stratification for $\mathrm{Var}(f) = Y \subseteq \mathbb{C}^n$, and let $\mathfrak{x}$ be a point in σ. Thom and Mather have shown that near $\mathfrak{x}$ there is a homeomorphism of germs between $(\mathbb{C}^n, \mathrm{Var}(f))$ and $(\mathbb{C} \times \mathbb{C}^{n-1}, \mathbb{C} \times \mathrm{Var}(g))$ where $g = g(x_2, \ldots, x_n)$ for suitable g.

(2) The homeomorphisms of the previous item can in general not be chosen differentiably: the divisors to $xy(x+y)(x+2y)$ and $xy(x+y)(x+zy)$ in $\mathbb{C}^3$ are locally topologically equivalent outside $z(z-1)(z-2)=0$. Yet, a differentiable isomorphism of the germs would induce a correspondence of their logarithmic stratifications, which is manifestly impossible along the z-axis.

(3) Let $Y \subseteq X = \mathbb{C}^n$ be Saito-holonomic and let $\mathfrak{x} \in \sigma$ be a point in a logarithmic stratum. Then the evaluation of $\mathrm{Der}_{X,\mathfrak{x}}(-\log Y)$ at $\mathfrak{x}$ spans the tangent space of σ at $\mathfrak{x}$. Note that in Example 2.4, the points where the logarithmic vector fields all vanish forms a positive-dimensional set, corresponding to a locally infinite stratification. The loci $\{\mathfrak{x} \in X \mid \mathrm{rk}_{\mathbb{C}}(\mathrm{Der}_X(-\log Y) \otimes_{\mathscr{O}_X} \mathscr{O}_{X,\mathfrak{x}}/\mathfrak{m}_{\mathfrak{x}}) = i\}$ are the candidates for logarithmic strata. They can be computed via Fitting ideals and are certified as holonomic strata if they have the expected dimension.

(4) Suppose Y is Saito-holonomic. Near $\mathfrak{x} \in \sigma$, consider the foliation of X obtained by integrating $\dim(\sigma)$ many independent elements δ_i in $\mathrm{Der}_X(-\log f)$ with $\delta_i \bullet (f) = 0$ that do not vanish near $\mathfrak{x}$ (any such collection is involutive). A choice of $n - \dim(\sigma)$ independent vector fields that are transversal to σ at $\mathfrak{x}$ induces a local analytic isomorphism between the pair (X, Y) and the product of $\mathbb{C}^{\dim \sigma}$ with a pair $(\mathbb{C}^{n-\dim(\sigma)} = X', Y')$ where the divisor $Y' \subseteq X'$ is a cross-section of Y transversal to σ at $\mathfrak{x}$. Compare [41, 3.5, 3.6].

(5) Whitney shows in [50] that the canonical Whitney stratification $\Sigma_{X,Y}$ is stable under all local analytic isomorphisms of X that fix Y. In the Saito-holonomic case, the local product structure implies that the logarithmic stratification refines the canonical Whitney stratification and is Whitney itself, compare [15, Prop. 3.11] and [16] for further details.

(6) To any ideal $\mathscr{I}_Y$ in $\mathscr{O}_X$ one can associate the (left) ideal in the sheaf of $\mathbb{C}$-linear differential operators $\mathscr{D}_X$ on X generated locally by the logarithmic derivations along Y. The cokernel $M^{\log Y}$ of this ideal may or may not be holonomic in the sense of Kashiwara, even for divisors. The notions of holonomicity are not the same: while Saito-holonomicity implies that $M^{\log Y}$ is a holonomic module in the free case [41, (3.18)], the example $(x-yz)(x^4+x^4y+y^4)$ shows that the implication cannot be reversed. $\diamond$

2.2. **Homogeneity conditions.**

Definition 2.7. Let $\mathfrak{x} \in X$.

- $f \in \mathscr{O}_{X,\mathfrak{x}}$ is *Euler-homogeneous* at $\mathfrak{x}$ if there is a vector field $E_{\mathfrak{x}}$ on $\mathscr{O}_{X,\mathfrak{x}}$ with $E_{\mathfrak{x}} \bullet (f) = f$. If $E_{\mathfrak{x}}$ can be chosen to vanish at $\mathfrak{x}$ then f is called *strongly Euler-homogeneous* at $\mathfrak{x}$.
- $f \in \mathscr{O}_X$ is (strongly) Euler-homogeneous if it is so at each $\mathfrak{x} \in \mathrm{Var}(f)$.
- A (not necessarily reduced) divisor $Y \subseteq X$ is (strongly) Euler-homogeneous, if near every $\mathfrak{x} \in Y$ there is a (strongly) Euler-homogeneous germ $f_{\mathfrak{x}} \in \mathscr{O}_{X,\mathfrak{x}}$ with $Y_{\mathfrak{x}} = \mathrm{Div}(f_{\mathfrak{x}})$.

Remark 2.8. (1) Any $f \in \mathscr{O}_X$ is strongly Euler-homogeneous in every smooth point of the associated reduced divisor.

(2) If $u \in \mathscr{O}_U$ is a unit then Euler-homogeneity is not always inherited from f to uf: $\exp(z) \cdot f(x,y)$ is Euler-homogeneous via ∂_z, but f has no reason to be so as well.

(3) In contrast, f is strongly Euler-homogeneous on U if and only uf is. Indeed, if $E_{\mathfrak{x}}$ is a strong Euler field for f then $\frac{uE_{\mathfrak{x}}}{u+E_{\mathfrak{x}}\bullet(u)}$ is one for uf.

(4) Suppose $Y \subseteq X$ is a divisor. If there is an analytic splitting $(X, Y) = \mathbb{C}\times(X', Y')$ then Y is strongly Euler-homogeneous if and only if Y' is, [28, Lem. 3.2]. On the other hand, Euler-homogeneity is not necessarily inherited from Y to Y': $\mathbb{C} \times Y'$ is always Euler-homogeneous thanks to the exponential function.

(5) Euler-homogeneity of f is an open condition, while strong Euler-homogeneity is not. For example, $f = zx^4 + xy^4 + y^5$ is strongly Euler-homogeneous at the origin, but it cannot be so along the z-axis because otherwise f should be in the ideal $(x, y, z - \lambda) \cdot (f_x, f_y, f_z)$, which it is not unless $\lambda = 0$.

(6) A strongly Euler-homogeneous divisor may fail to be Saito holonomic: consider $f = xy(x + y)(x + zy)$. Indeed, along the z-axis, f has the strong Euler derivation $(x\partial_x + y\partial_y)/4$; along the y-axis and along the line $x + y = z - 1 = 0$, f has the strong Euler derivation $(x + zy)\partial_z$; in all other places $\operatorname{Var}(f)$ is smooth. On the other hand, every point on the z-axis is a logarithmic stratum.

(7) If f is strongly Euler-homogeneous and Saito-holonomic at $\mathfrak{x}$, $\mathfrak{x}$ a point in the stratum σ, 2.6.(4) yields a splitting where f is constant along $\dim(\sigma)$ directions of X', and the transversal section is strongly Euler-homogeneous. ⋄

Definition 2.9. For $f \in \mathscr{O}_X$ we set

$$\operatorname{Der}_X(-\log_0 f) := \operatorname{Der}_X(-\log f) \cap \operatorname{ann}_{\mathscr{D}_X}(f^s) = \{\delta \in \operatorname{Der}_X(-\log f) \mid \delta \bullet (f) = 0\}.$$

Remark 2.10. (1) Geometrically, elements of $\operatorname{Der}_X(-\log f)$ are tangent to the hypersurface $\operatorname{Var}(f)$ while those in $\operatorname{Der}_X(-\log_0 f)$ are tangent to all level hypersurfaces of f.

(2) Let x, x' be two local coordinate systems at $\mathfrak{x} \in X$. Then the gradients $\nabla_x(f)$ and $\nabla_{x'}(f)$ differ by the Jacobian matrix: $\nabla_x(f) = (\nabla_{x'}(f))\cdot((\frac{\partial x'}{\partial x}))$. Hence, $\operatorname{Der}_X(-\log_0 f)$ varies in a dual fashion with the coordinate system.

(3) If u is a local unit, $E_\mathfrak{x}$ a strong Euler-homogeneity for $f \in \mathscr{O}_{X,\mathfrak{x}}$ and $\delta \in \operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f)$ then $\delta - \frac{\delta\bullet(u)}{u+E_\mathfrak{x}\bullet(u)}E_\mathfrak{x} \in \operatorname{Der}_{X,\mathfrak{x}}(-\log_0(uf))$; this association is an $\mathscr{O}_{X,\mathfrak{x}}$-module isomorphism.

(4) If f is Euler-homogeneous at $\mathfrak{x}$, then the maps

$$\operatorname{Der}_{X,\mathfrak{x}}(-\log f) \ni \delta \mapsto \delta - \frac{\delta \bullet (f)}{f} \cdot E_\mathfrak{x} \in \operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f), \tag{2.1}$$

$$\operatorname{Der}_{X,\mathfrak{x}}(-\log f) \ni \delta \mapsto \frac{\delta \bullet (f)}{f} \cdot E_\mathfrak{x} \in \mathscr{O}_{X,\mathfrak{x}} \cdot E_\mathfrak{x} \tag{2.2}$$

give a split exact sequence

$$0 \longrightarrow \operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f) \longrightarrow \operatorname{Der}_{X,\mathfrak{x}}(-\log f) \longrightarrow \mathscr{O}_{X,\mathfrak{x}} \cdot E_\mathfrak{x} \longrightarrow 0.$$

These splittings depend on the choice of the equation f for the divisor. ⋄

Remark 2.11. If $Y \subseteq X$ are algebraic, then logarithmic vector fields and differentials, as well as differential annihilators can be defined in both the analytic and the algebraic category. Since they are all defined by syzygies between derivatives of f, the analytic objects are the pullbacks to the analytic category of the algebraic objects. Questions about their generation and homological properties can hence be investigated on either side. ⋄

3. The Liouville complex

As always, X is a complex analytic manifold. Let f be a global section of $\mathscr{O}_X$. We consider the principal $\mathscr{D}_X[s]$-module $\mathscr{D}_X[s] \bullet f^s$ generated by the symbol f^s subject to local relations spelled out in (1.1). Whenever f is Euler-homogeneous, $\mathscr{D}_X[s] \bullet f^s$ and $\mathscr{D}_X \bullet f^s$ agree. In this section we determine the structure of $\operatorname{ann}_{\mathscr{D}_X[s]}(f^s)$ for a large family of divisors with good homogeneity conditions by computing its characteristic cycle.

The Liouville ideal. For any filtration F on a ring A we denote by $\operatorname{gr}_F(A)$ the associated graded object, and by $\operatorname{gr}_F(a)$ the image of $a \in A$ in $\operatorname{gr}_F(A)$. The order filtration on $\mathscr{D}_X$ leads to a sheaf of $(0,1)$-graded commutative rings $\operatorname{gr}_{(0,1)}(\mathscr{D}_X)$ whose sections are naturally identified with the functions on T^*X that are polynomial in the cotangent directions.

Definition 3.1. Let the *Liouville ideal* be the ideal $\mathscr{L}_f \subseteq \operatorname{gr}_{(0,1)}(\mathscr{D}_X)$ generated by the $(0,1)$-symbols (that is, the lead terms under the order filtration) of $\operatorname{Der}_X(-\log_0 f) \subseteq \operatorname{ann}_{\mathscr{D}_X[s]}(f^s)$:

$$\mathscr{L}_f = \operatorname{gr}_{(0,1)}(\mathscr{D}_X) \cdot \operatorname{gr}_{(0,1)}(\operatorname{Der}_X(-\log_0 f)).$$

Remark 3.2. Suppose f is Euler-homogeneous at $\mathfrak{x} \in X$: $E_\mathfrak{x}(f) = f$. In this remark, we read derivations as formal $\mathscr{O}_X$-linear combinations on the derivatives $f_i = \partial_i \bullet (f)$. In this sense, elements of $\operatorname{Der}_X(-\log_0 f)$ are syzygies.

(1) Let x, x' be two coordinate systems with (column) vectors of partial differentiation operators ∂, ∂'. Denote c_δ, c'_δ the coefficient (column) vectors of a derivation $\delta = c_\delta^T \cdot \partial = {c'_\delta}^T \cdot \partial'$ in the two coordinate systems. If $J = ((\partial x'_j/\partial x_i))$ is the Jacobian matrix, $\partial = J \cdot \partial'$ and $c'_\delta = J^T \cdot c_\delta$. In particular, $\operatorname{Der}_X(-\log_0 f) = J^T \cdot \operatorname{Der}'_X(-\log_0 f)$ and so $\mathscr{L}_f$ is well-defined.

(2) Let $u \in \mathscr{O}_{X,\mathfrak{x}}$ be a unit. The $\mathscr{O}_{X,\mathfrak{x}}[s]$-algebra automorphism $\partial_i \longrightarrow \partial_i - \frac{\partial_i \bullet (u)}{u} s$ on $\mathscr{D}_{X,\mathfrak{x}}[s]$ identifies $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[s]}(f^s)$ with $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[s]}(u^s f^s)$. On the level of derivations, if there is a strong Euler field $E_\mathfrak{x}$ for f at $\mathfrak{x}$, this corresponds to an $\mathscr{O}_{X,\mathfrak{x}}$-isomorphism $\alpha_u\colon \operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f) \longrightarrow \operatorname{Der}_{X,\mathfrak{x}}(-\log_0(uf))$ via $\delta \mapsto \delta - \frac{\delta \bullet (u)}{u + E_\mathfrak{x} \bullet (u)} E_\mathfrak{x}$ from Remark 2.10.

(3) If $\nabla(u), \nabla'(u)$ are the gradient (column) vectors of u in the two coordinate systems, $\nabla(u) = J \cdot \nabla'(u)$. The $\mathscr{O}_{X,\mathfrak{x}}$-isomorphism α_u from the previous item sends $c_\delta^T \cdot \partial$ to $c_\delta^T \cdot (I_n - \frac{\nabla(u) \cdot c_{E_\mathfrak{x}}^T}{u + E_\mathfrak{x} \bullet (u)}) \cdot \partial$, where I_n is the identity matrix. On the other hand, in x'-coordinates, $c_\delta^T \cdot \partial = c_\delta^T \cdot J \cdot \partial'$ is sent to $(c_\delta^T \cdot J)(I_n - \frac{\nabla'(u) \cdot {c'}_{E_\mathfrak{x}}^T}{u + E_\mathfrak{x} \bullet (u)})\partial'$. A simple calculation shows now that α_u commutes with coordinate changes.

(4) One may consider the symbols $y_f = \operatorname{gr}_{(0,1)}(\partial)$ as indeterminates over $\mathscr{O}_X$, labeled by the choice of the defining equation f for a fixed strongly Euler-homogeneous divisor. Locally, the maps α_u allow to identify these symbols, respecting the action by the Jacobian under a coordinate change. With $(y_{uf})^T (I_n - \frac{\nabla'(u) \cdot {c'_{E_\mathfrak{x}}}^T}{u + E_\mathfrak{x} \bullet (u)}) = y_f^T$ there is a local $\mathscr{O}_X$-isomorphism of $\operatorname{gr}_{(0,1)}(\mathscr{D}_X)$ sending $\operatorname{gr}_{(0,1)}(\delta) \mapsto \operatorname{gr}_{(0,1)}(\delta) - \frac{\delta \bullet (u)}{u + E_\mathfrak{x} \bullet (u)} \operatorname{gr}_{(0,1)}(E_\mathfrak{x})$ that commutes with coordinate changes, patches over the domain where $E_\mathfrak{x}$ and u are defined, and sends $\mathscr{L}_f$ to $\mathscr{L}_{uf}$.

It follows that if Y is a strongly Euler-homogeneous divisor then the local geometric and algebraic properties of $\mathscr{L}_Y$ are independent of the choice of the local equation cutting out Y.

(5) For any coherent $\mathscr{D}_X$-module $\mathscr{M}$, filtered compatibly with the order filtration on $\mathscr{D}_X$, the support in the cotangent bundle T^*X of the associated graded object $\operatorname{gr}_{(0,1)}(\mathscr{M})$ is the *characteristic variety* $\operatorname{charV}(\mathscr{M})$.

Generically (in X), $\mathscr{L}_f$ is of height $n-1$ in $\mathrm{gr}_{(0,1)}(\mathscr{D}_X)$ since $\mathrm{Der}_X(-\log_0 f)$ is of rank $n-1$. Wherever some derivative f_i is nonzero, the ideal $\mathscr{L}_f$ is cut out by $\{y_j - \frac{f_j}{f_i} y_i\}$. In fact, (see, e.g., [36]) over the smooth locus of f the characteristic variety of $\mathscr{D}_X \bullet (f^s)$ is locally a smooth complete intersection of codimension $n-1$, and cut out by $\mathscr{L}_f$. ◇

3.1. **The complex.** Any symplectic manifold X admits a one-form on the cotangent bundle whose differential is the symplectic form. Any such one-form, not necessarily unique, is called a *symplectic potential*. Let X be a complex analytic manifold; it then has a natural symplectic structure. The *Liouville form* on X is the unique symplectic potential on X that in local (Darboux) coordinates takes the form

$$y\,\mathrm{d}x := y_1 \mathrm{d}x_1 + \ldots + y_n \mathrm{d}x_n$$

with base coordinates x and corresponding cotangent coordinates y.

As functions on the cotangent bundle fibers, the y_i can be identified with the symbols of ∂_i in the corresponding coordinate system. This makes it clear that $y\,\mathrm{d}x$ is independent of the coordinate system, since ∂_i and $\mathrm{d}x_i$ vary dually with any coordinate transformation.

For any $\mathscr{O}_X$-module $\mathscr{M}$, write

$$\mathscr{M}[y] := \pi_* \pi^* \mathscr{M} = \mathscr{M} \otimes_{\mathscr{O}_X} \pi_*(\mathscr{O}_{T^*X})$$

where $\pi \colon T^*X \longrightarrow X$ is the natural projection from the cotangent bundle.

Definition 3.3. Suppose f is a global non-constant function on X. The modules of the *Liouville complex* $C_f^\bullet$ *of* f are defined by

$$\Omega_X^n \otimes_{\mathscr{O}_X} \frac{1}{f} \mathscr{O}_X \otimes_{\mathscr{O}_X} C_f^i = \Omega_X^i(\log_0 f)[y] \subseteq \frac{1}{f} \Omega_X^i[y],$$

i.e., up to twist by $1/f$ they are the kernels of $\mathrm{d}f \wedge (-)$ on $\left(\omega_X^{-1} \otimes_{\mathscr{O}_X} \Omega_X^\bullet\right)[y]$ where ω_X^{-1} is the inverse of the invertible sheaf Ω_X^n. The differential of $C_f^\bullet$ is given by the exterior product with the Liouville form $y\,\mathrm{d}x$.

Remark 3.4. (1) Since $\mathrm{d}f \wedge y\,\mathrm{d}x = -y\,\mathrm{d}x \wedge \mathrm{d}f$, $C_f^\bullet$ is indeed a complex in the category of $\mathscr{O}_X[y]$-modules.

(2) Locally, $(C_f^\bullet, y\,\mathrm{d}x)$ can be identified with $(\Omega_X^\bullet(\log_0 f)[y], y\,\mathrm{d}x)$. On $X = \mathbb{C}^n$, with a chosen coordinate system x and corresponding derivations ∂, identify y with the order symbols of ∂, and $\mathrm{gr}_{(0,1)}(\mathscr{D}_X)$ with $\mathscr{O}_X[y]$. Denote by $K^\bullet$ the cohomological Koszul complex of the regular sequence y on $\frac{1}{f}\mathscr{O}_X[y]$ and identify $K^i = \frac{1}{f} \bigwedge^{n-i} (\mathscr{O}_X[y])^n$ with $\frac{1}{f}\Omega_X^i[y]$ in the usual way. Then $C_f^\bullet$ is isomorphic to the restriction of $K^\bullet$ to the complex whose modules are the kernels in the Koszul complex on $\mathscr{O}_X[y]$ induced by $f_1 = \partial_1(f), \ldots, f_n = \partial_n(f)$. As such, locally in a chosen coordinate system, the Liouville complex is isomorphic to the approximation complex $\mathscr{C}$ from [31] and we use some of their techniques.

(3) The induced y-grading on $\mathrm{gr}_{(0,1)}(\mathscr{D}_X) \cong \mathscr{O}_X[y]$ makes $C_f^\bullet$ graded if we position $\mathscr{O}_X \subseteq C_f^n$ in degree zero, and more generally the elements of $f \cdot \omega_X^{-1} \otimes_{\mathscr{O}_X} \Omega_X^i(\log f) \subseteq f \cdot \omega_X^{-1} \otimes_{\mathscr{O}_X} \Omega_X^i(\log f)[y] = C_f^i$ in degree $n-i$. This agrees with the intrinsic y-grading from the definition.

We write $C^i_{f,\mathfrak{x}}$ for the stalk of C^i_f at $\mathfrak{x} \in X$ (that is, we tensor with $\mathscr{O}_{X,\mathfrak{x}}$) and denote the part of C^i_f in degree j by $(C^i_f)_j$,

$$C^i_f = \bigoplus (C^i_f)_j.$$

We have $\operatorname{pdim}_{\mathscr{O}_{X,\mathfrak{x}}[y]}(C^i_{f,\mathfrak{x}}) = \operatorname{pdim}_{\mathscr{O}_{X,\mathfrak{x}}}(C^i_{f,\mathfrak{x}})_j$ whenever $(C^i_{f,\mathfrak{x}})_j \neq 0$ since

$$C^i_f = (C^i_f)_{n-i} \otimes_{\mathscr{O}_X} \operatorname{gr}_{(0,1)}(\mathscr{D}_X).$$

(4) In each stalk on X, the module $(C^i_f)_{n-i}$ is, up to a free summand, a second syzygy (of the cokernel of $\mathrm{d}f\wedge$). By [26, Thm. 3.6], the property of $(C^i_f)_{n-i}$ of being a second syzygy over $\mathscr{O}_X$ is equivalent to it being $\mathscr{O}_X$-reflexive since a regular ring is a normal domain. Being a second syzygy also forces the $\mathscr{O}_X$-projective dimension of $(C^i_f)_{n-i}$ to be at most $n-2$. Both properties percolate to C^i_f.

(5) The isomorphism

$$\Omega^n \otimes_{\mathscr{O}_X} \frac{1}{f}\mathscr{O}_X \otimes_{\mathscr{O}_X} \operatorname{Der}_X(-\log_0 f) \ni \frac{\mathrm{d}x}{f} \otimes \sum_{i=1}^n a_i\partial_i \longleftrightarrow \frac{\sum_{i=1}^n (-1)^i a_i \widehat{\mathrm{d}x_i}}{f} \in \Omega_X^{n-1}(\log_0 f)$$

shows that $\mathscr{O}_X[y]/\mathscr{L}_f$ is the terminal cohomology group of $C^\bullet_f$.

(6) In general, one has $\Omega^i_X(\log f) = \left(\bigwedge^i(\operatorname{Der}_X(-\log f))^*\right)^{**}$, where the star denotes $\operatorname{Hom}_{\mathscr{O}_X}(-, \Omega^n_X(\log f))$. If f is strongly Euler-homogeneous (or more generally if E is an Euler field for f with $u + E \bullet (u)$ nonzero), Remark 2.10 implies that there is an $\mathscr{O}_X[y]$-automorphism of $\operatorname{gr}_{(0,1)}(\mathscr{D}_X)$ that carries $\Omega^i_X(\log_0(f))[y]$ into $\Omega^i_X(\log_0(uf))[y]$. In particular, $C^\bullet_f$ and $C^\bullet_{uf}$ have the same algebraic properties. $\diamond$

3.2. **Exactness.**

Notation 3.5. Throughout we use the word *resolution* to denote a finite complex with a unique cohomology group (at its end). No specific properties of the modules involved (such as freeness, projectivity or injectivity) are implied.

We show here that $C^\bullet_f$ is a resolution of $\mathscr{L}_f$ when $n \leq 3$ or if the modules $\Omega^i_X(\log f)$ have locally high depth over $\mathscr{O}_X$. For this, we need to slightly generalize the construction of $C^\bullet_f$.

Notation 3.6. Let $\phi = \phi_1, \dots, \phi_n$ be in $\mathscr{O}_X$, X smooth and affine. Then let $C^\bullet_\phi$ be the (graded) subcomplex of $(\Omega^\bullet_X[y], y\,\mathrm{d}x\wedge)$ given by the kernels of $\phi \wedge (-)$, where we read ϕ as the coefficients of a one-form. One can, as for $C^\bullet_f$, ask whether it is a resolution.

Lemma 3.7. *The complexes $C^\bullet_\phi$ and $C^\bullet_{g\phi}$ agree for all nonzero $g \in \mathscr{O}_X$. If ϕ generates an ideal of height at least two at $\mathfrak{x} \in X$ then $C^\bullet_\phi$ is exact at $\mathfrak{x}$ for all $i \leq \min(\operatorname{ht}_\mathfrak{x}(\phi), n-1)$ where $\operatorname{ht}_\mathfrak{x}(\phi)$ is the height of the ideal generated by ϕ in $\mathscr{O}_{X,\mathfrak{x}}$.*

Proof. The first part is clear as we calculate in a domain.

For the second part, suppose $\beta \wedge \phi = \beta \wedge y\,\mathrm{d}x = 0$, $\beta \in \Omega^i_X[y]$, $i \leq n-1$. Then $\beta = y\,\mathrm{d}x \wedge \gamma$ for $\gamma \in \Omega^{i-1}_X[y]$. Then $\beta' := \gamma \wedge \phi \in C^i_\phi$ and $\beta' \wedge y\,\mathrm{d}x = 0$. One checks that this induces an $\mathscr{O}_X[y]$-endomorphism of $H^i(C^\bullet_\phi)$ sending the class of β to the class of β' and which we denote $(-)'_\phi$.

Now $H^i(C^\bullet_\phi)$ is graded in y and $(-)'_\phi$ reduces degree by one. Thus, eventually $\beta^{(k)} := (\beta^{(k-1)})'$ will be the zero class and so $\beta^{(k)} = \gamma^{(k)} \wedge y\,\mathrm{d}x$ with $\gamma^{(k)} \wedge \phi = 0$.

In a regular local ring, $\mathrm{ht}_{\mathfrak{r}}(\phi)$ is the largest i such that the Koszul cocomplex of ϕ on $\mathscr{O}_X$ is exact in positions lower than i. Since $i \le \mathrm{ht}_{\mathfrak{r}}(\phi)$, $\gamma^{(k)} = \delta^{(k)} \wedge \phi$ for some $\delta^{(k)} \in \Omega_X^{i-2}[y]$. Then $-\delta^{(k)} \wedge y\,\mathrm{d}x + \gamma^{(k-1)}$ is in the kernel of ϕ and, like $\gamma^{(k-1)}$, multiplies against $y\,\mathrm{d}x$ to $\beta^{(k-1)}$. So, if the class of $\beta^{(k)}$ in $H^i(C_\phi^\bullet)$ is zero (that is, if $\beta^{(k)}$ is image under $\wedge y\,\mathrm{d}x$ in $C_\phi^\bullet$) then the same is true for $\beta^{(k-1)}$, and thus eventually of $\beta = \beta^{(0)}$. □

We make precise in the following definition the homological conditions we require from our divisors; U is an open set in X.

Definition 3.8. The section $f \in \mathscr{O}_U$ is *tame* if for all i the projective dimension of $\Omega_U^i(\log f)$ over $\mathscr{O}_U$ is at most i in each stalk. A divisor Y is tame if it allows tame defining equations locally everywhere.

Theorem 3.9. *For $f \in \Gamma(X, \mathscr{O}_X)$, suppose that either*

- *$n \le 3$, or*
- *f is strongly Euler-homogeneous, Saito-holonomic, and tame.*

Then the Liouville complex $C_f^\bullet$ is a resolution of the Liouville ideal $\mathscr{L}_f$.

Before we embark on the proof, inspired by [31], we collect some helpful facts.

Notation 3.10. If δ is a vector field and ω a differential form on an open set $U \subseteq X$ then we denote by $\delta \diamond \omega$ the contraction of ω along δ. In particular, on the level of 1-forms, $\diamond$ denotes the natural pairing between TX and T^*X with values in the 0-forms $\mathscr{O}_X$.

Contraction of $\Omega_X^i(\log f)$ by a logarithmic derivation induces an $\mathscr{O}_X$-morphism to $\Omega_X^{i-1}(\log f)$. In particular, for an Euler field E_U for f on the open set $U \subseteq X$ and for $\alpha \in \Omega_U^i(\log f)$,

$$\alpha \quad = \quad \alpha \wedge \left(E_U \diamond (\frac{\mathrm{d}f}{f}) \right) = \left(E_U \diamond (\alpha \wedge \frac{\mathrm{d}f}{f}) - (E_U \diamond \alpha) \wedge \frac{\mathrm{d}f}{f} \right) \cdot (-1)^i. \tag{3.1}$$

Note that $\{\omega/f \in \frac{1}{f}\Omega_U^i \mid \omega \wedge \mathrm{d}f = 0\} = \{\omega/f \in \Omega_U^i(\log f) \mid \omega \wedge \mathrm{d}f = 0\}$ and define a sheaf $\Omega_X^\bullet(\log_E f)$ by

$$\Omega_U^i(\log_E f) = E_U \diamond \left(\Omega_U^{i+1}(\log_0 f)\right),$$

a submodule of $\Omega_X^i(\log f)$, well-defined for the same reasons that make $\mathscr{L}_f$ independent of the choice of the coordinate system: the Jacobian matrix acts (via exterior powers) on Ω_U^i and (dually) on $y\,\mathrm{d}x$.

Lemma 3.11. *Suppose that $f \in \mathscr{O}_X$ has a global Euler field. The contraction $E\colon \Omega_X^i(\log f) \longrightarrow \Omega_X^{i-1}(\log f)$ is injective on $\Omega_X^i(\log_0 f)$, and*

$$\Omega_X^\bullet(\log f) = \Omega_X^\bullet(\log_0 f) \oplus \Omega_X^\bullet(\log_E f) \cong \Omega_X^\bullet(\log_0 f) \oplus \Omega_X^{\bullet+1}(\log_0 f).$$

Proof. The second term in the difference (3.1) of α is in $\Omega_X^i(\log_0 f)$. As $\alpha \wedge \frac{\mathrm{d}f}{f}$ is in $\Omega_X^{i+1}(\log_0 f)$, the first term of the difference is in $\Omega_X^i(\log_E f)$. Thus, $\Omega_X^i(\log f) = \Omega_X^i(\log_0 f) + \Omega_X^i(\log_E f)$. If $\alpha \wedge \frac{\mathrm{d}f}{f} = 0$ and $E \diamond \alpha = 0$ for $\alpha \in \Omega_X^i(\log f)$ then (3.1) yields $\alpha = 0$. So the sum is direct.

If $0 \ne \alpha \in \Omega_X^i(\log_0 f)$, then by (3.1) α is $(E \diamond \alpha) \wedge \frac{\mathrm{d}f}{f}$ up to sign, and so surely $E \diamond \alpha$ is nonzero. □

Remark 3.12. (1) If f permits a global Euler field, then $C^i_{f,\mathfrak{x}}$ is locally a summand of both $\Omega^i_{X,\mathfrak{x}}(\log f)[y]$ and $\Omega^{i-1}_{X,\mathfrak{x}}(\log f)[y]$, and so (cf. Remark 3.4, parts (3) and (4))

$$\mathrm{pdim}_{\mathscr{O}_{X,\mathfrak{x}}[y]}(C^i_{f,\mathfrak{x}}) \leq \min\{\mathrm{pdim}_{\mathscr{O}_{X,\mathfrak{x}}} \Omega^i_{X,\mathfrak{x}}(\log f), \mathrm{pdim}_{\mathscr{O}_{X,\mathfrak{x}}} \Omega^{i-1}_{X,\mathfrak{x}}(\log f)\} \leq n-2.$$

(2) The complexes $C^\bullet_f$ and $C^\bullet_{f^k}$ are isomorphic up to a shift by kf^{k-1}.

(3) If $X = \mathbb{C} \times X'$ and f does not depend on x_1 then $C^\bullet_f$ is locally isomorphic to the Koszul cocomplex of y_1 on $\mathscr{O}_X[y_1]$ tensored with the Liouville complex $C^\bullet_{f_{|_{X'}}}$ to the restriction of f to X'.

(4) In a smooth point $\mathfrak{x}$ of the reduced hypersurface $\mathrm{Var}(f_{\mathrm{red}})$, a suitable analytic coordinate change arranges that $f = {x'_n}^k$. In new coordinates, $\mathrm{d}f$ is $k \cdot {x'_n}^{k-1}\mathrm{d}x'_n$ and so $C^\bullet_{f,\mathfrak{x}}$ is (essentially) the Koszul complex in the indeterminates $y'_1, \ldots, y'_{n-1}$ on $\mathscr{O}_{X,\mathfrak{x}}[y]$. In particular, $C^\bullet_{f,\mathfrak{x}}$ is a resolution in $\mathfrak{x}$.

(5) If f is Euler-homogeneous the previous item shows that $C^\bullet_f$ is exact outside the singular locus of the hypersurface f_{red}.

(6) Over a domain, the wedge product of two 1-forms is zero if and only if they are proportional. Thus, $C^0_f = 0$ and $C^1_f = \ker(\frac{1}{f}\Omega^1_X[y] \xrightarrow{\frac{\mathrm{d}f}{f}\wedge} \frac{1}{f}\Omega^2_X[y])$ is the free module $\mathscr{O}_X[y]\cdot\frac{f_{\mathrm{red}}\mathrm{d}f}{f^2}$. As $y\,\mathrm{d}x$ and $\frac{\mathrm{d}f}{f}$ are generically independent, $y\,\mathrm{d}x\colon C^1_f \longrightarrow C^2_f$ is injective. ⋄

Notation 3.13. Let $\mathfrak{x} \in X$. For the rest of this section, put $R = \mathscr{O}_{X,\mathfrak{x}}$ and $S = \mathrm{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}}) \cong R[y]$, a standard graded polynomial ring over a Noetherian regular local ring.

We shall make use of the Acyclicity Lemma of Peskine–Szpiro from [40]:

Theorem 3.14. *Let $0 \longrightarrow \Lambda^0 \longrightarrow \cdots \longrightarrow \Lambda^t$ be a complex in the category of finitely generated modules over the Noetherian ring A such that $\mathrm{pdim}(\Lambda^i) \leq i$, and such that the non-vanishing cohomology modules $H^i(\Lambda^\bullet)$ have depth zero for $i < t$. If $t \leq \mathrm{depth}(A)$ then the complex is a resolution of its terminal cohomology group.* □

Lemma 3.15. *The Liouville complex is a reflexive resolution of $\mathscr{L}_f$ if $n \leq 3$.*

Proof. If $n < 3$ then the claim follows from Remark 3.12.

Let $f_{\mathrm{nil}} = \gcd(f_1, \ldots, f_n)$ and $\phi = \{f_1/f_{\mathrm{nil}}, \ldots, f_n/f_{\mathrm{nil}}\}$, recalling that f_i denotes $\partial_i \bullet (f)$. Then $C^\bullet_f$ and $C^\bullet_\phi$ are equal up to a twist. By construction, in each stalk the ideal generated by ϕ is either the unit ideal or has height at least 2.

If ϕ generates the unit ideal at $\mathfrak{x} \in X$, then a base change produces $\phi = (1, 0, \ldots, 0)$, so that $C^\bullet_\phi$ is essentially the Koszul cocomplex on $y_2, \ldots, y_n$. If, on the other hand, the ideal generated by ϕ has height at least 2 then $H^i(C^\bullet_f) = 0$ for $i \leq 2$ by Lemma 3.7. □

In higher dimension, strongly Euler-homogeneous Saito-holonomic divisors permit a general method of arguing that we use to prove Theorem 3.9.

Proof of Theorem 3.9. We shall proceed by induction on n. Because of Lemma 3.15 we can assume that $n \geq 4$, and that f is tame, strongly Euler-homogeneous and Saito-holonomic.

Outside $\mathrm{Var}(f)$, $C^\bullet_f$ is exact by Remark 3.12. Let $\mathfrak{x} \in X$ be in a positive-dimensional logarithmic stratum σ of $\mathrm{Var}(f)$. By Remarks 2.6 and 2.8.(7), there is an analytic coordinate change $x \rightsquigarrow x'$ near $\mathfrak{x}$ transforming a neighborhood of $\mathfrak{x}$ into $\mathbb{C}^{\dim\sigma}\times\mathbb{C}^{n-\dim\sigma}$ such that f is constant on the first factor. Let f' be the restriction of f to the second factor in the new coordinates, and denote by $y \rightsquigarrow y'$ the corresponding coordinate change in the cotangent variables. Then by Remark 3.12.(3), near $\mathfrak{x}$ the complex $C_f^\bullet$ is (relative to the induced $\operatorname{gr}_{(0,1)}(\mathscr{D}_X)$-isomorphism from Remark 3.2) isomorphic to the Koszul complex in $y'_1,\ldots,y'_{\dim\sigma}$, tensored with the Liouville complex of f'. Tameness (since $\Omega^\bullet_X(\log f) = \Omega^\bullet_{X'}(\log f')\otimes_{\mathbb{C}}\Omega^\bullet_{\mathbb{C}^{\dim(\sigma)}}$), strong Euler-homogeneity (Remark 2.8), and Saito holonomicity (from the definition) are inherited from f to f'. Since f' uses fewer than n coordinates, the inductive hypothesis assures that the Liouville complex of f' is a reflexive resolution of the residue ring of $\mathscr{L}_{f'}$. Since $C^\bullet_{f'}$ is y'_i-torsion-free for $i\le\dim\sigma$, $C^\bullet_f$ is a reflexive resolution of $\mathscr{L}_f$ at $\mathfrak{x}$.

It follows that $H^{<n}(C^\bullet_f)$ is supported at the zero-dimensional strata of the logarithmic stratification of f. Then, for all j, the graded components $(C^\bullet_{f,\mathfrak{x}})_j$ are complexes of finitely generated $\mathscr{O}_{X,\mathfrak{x}}$-modules whose i-th cohomology is zero-dimensional for $i<n$. Since f is tame, $\operatorname{pdim}_R((C^{n-i}_{f,\mathfrak{x}})_j)\le i$ for all i by Remarks 3.12.(1) and 3.4.(3) and so $C^\bullet_f$ is, by the Acyclicity Lemma, a resolution of $\mathscr{O}_{X,\mathfrak{x}}/\mathscr{L}_f$ everywhere. □

3.3. **Cohen–Macaulayness.**

Theorem 3.16. *Let X be a complex manifold, and let Y be a strongly Euler-homogeneous and Saito-holonomic divisor cut out by the global equation $f\in\mathscr{O}_X$. Consider the following conditions:*

(1) f is tame: $\operatorname{pdim}_{\mathscr{O}_X}\Omega^i_X(\log Y)\le i$ for all $i\ge 0$.
(2) The quotient $\mathscr{O}_X[y]/\mathscr{L}_Y$ is Cohen–Macaulay.

Then (1) implies (2) in any case, and if the Liouville complex $C^\bullet_f$ is exact then (2) implies (1).

Proof. For strongly Euler-homogeneous divisors, homological properties of $\mathscr{L}_f$ depend only on the divisor and not its defining equation (Remarks 2.10, 3.2). Tameness, Cohen–Macaulayness and exactness of the Liouville complex are all local properties. We may hence calculate in the ring $\mathscr{O}_{X,\mathfrak{x}}[y]$. So, assume that on the open affine set $U\ni\mathfrak{x}$ there is a global section f defining Y that permits a strong Euler field.

Since generically $\mathscr{L}_f$ is a prime of height $n-1$, Cohen–Macaulayness is equivalent to $\operatorname{pdim}(\mathscr{L}_f)=n-1$. Picture $C^\bullet_{f,\mathfrak{x}}\cong(\Omega^\bullet_{X,\mathfrak{x}}(\log_0 f)[y], y\,\mathrm{d}x)$ as a row complex. Let $F^i_\bullet$ be a minimal graded free $\mathscr{O}_{X,\mathfrak{x}}[y]$-resolution (a column oriented downwards) of $C^i_{f,\mathfrak{x}}$, and let $F^\bullet_\bullet$ be a y-graded double complex that results from lifting the maps $y\,\mathrm{d}x\colon C^i_{f,\mathfrak{x}}\longrightarrow C^{i+1}_{f,\mathfrak{x}}$. The ring $\mathscr{O}_{X,\mathfrak{x}}$ is regular, the complex $C^\bullet_f$ is y-graded, and $y\,\mathrm{d}x$ has positive degree. Hence, the total complex to $F^\bullet_\bullet$ is minimal. As this total complex resolves $C^\bullet_{f,\mathfrak{x}}$ under the given hypotheses, by Theorem 3.9 it minimally resolves $\mathscr{O}_{X,\mathfrak{x}}[y]/\mathscr{L}_f$, positioned in cohomological degree n.

Suppose first that f is tame. Since $\Omega^i_{X,\mathfrak{x}}(\log f)\cong\Omega^i_{X,\mathfrak{x}}(\log_0 f)\oplus\Omega^{i+1}_{X,\mathfrak{x}}(\log_0 f)$, tameness implies that the length of $F^{i+1}_\bullet$ is at most $\operatorname{pdim}_{\mathscr{O}_{X,\mathfrak{x}}}(\Omega^{i+1}_{X,\mathfrak{x}}(\log_0 f))\le\operatorname{pdim}_{\mathscr{O}_{X,\mathfrak{x}}}(\Omega^i_{X,\mathfrak{x}}(\log f))\le i$. Thus, the total complex of $F^\bullet_\bullet$ is of length at most $n-1$ and hence $\mathscr{O}_{X,\mathfrak{x}}[y]/\mathscr{L}_f$ is Cohen–Macaulay.

Conversely, suppose that f is not tame but $C^\bullet_{f,\mathfrak{x}}$ is exact. Then the total complex of $F^\bullet_\bullet$ is minimal and free, resolves $\mathscr{O}_{X,\mathfrak{x}}/\mathscr{L}_f$ but has length greater than $n-1$. So $\mathscr{L}_f$ cannot be a Cohen–Macaulay ideal at $\mathfrak{x}$. $\square$

3.4. **Integrality.** In general, one has inclusions of $\mathscr{O}_{X,\mathfrak{x}}[y]$-ideals

$$\operatorname{gr}_{(0,1)}(\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}} f^s) \supseteq \operatorname{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}} \cdot \operatorname{Der}_X(-\log_0 f)) \supseteq (\mathscr{L}_f)_{\mathfrak{x}}. \tag{3.2}$$

In a smooth point of f_{red}, $\operatorname{gr}_{(0,1)}(\operatorname{ann}_{\mathscr{D}_X} f^s)$ is prime of height $n-1$. Hence, wherever $\mathscr{L}_f$ locally contains a prime ideal of height $n-1$, all three ideals are identical. This makes it very useful to know criteria that assure that $\mathscr{L}_f$ is prime, compare Theorem 3.26 below.

For example, if in local coordinates $f \in \mathscr{O}_U$ can be written as $f = u\prod_{i=1}^n x_i^{\alpha_i}$ where u is a unit and $\alpha_i > 0$ exactly when $1 \le i \le k$ then $\operatorname{Der}_U(-\log f)$ is the locally free module generated locally by the vector fields $\{(\alpha_j + \frac{x_j\partial_j \bullet(u)}{u})x_1\partial_1 - (\alpha_1 + \frac{x_1\partial_1\bullet(u)}{u})x_j\partial_j\}_{2\le j\le n}$. In particular, $\mathscr{L}_f$ is prime on U and the previous paragraph applies: over all normal crossing points of f, all three ideals in (3.2) are one and the same complete intersection prime ideal of height $n-1$. The same argument has been made more generally wherever f is free and quasi-homogeneous in [9].

Theorem 3.17. *Let $f \in \mathscr{O}_X$ be strongly Euler-homogeneous, Saito-holonomic, and tame. Then $\operatorname{gr}_{(0,1)}(\mathscr{D}_X)/\mathscr{L}_f$ is an integral domain.*

Proof. Again, we argue locally. By the discussion above, $\mathscr{L}_f$ is prime away from singular points of f_{red}. We argue by induction on the strata of the logarithmic stratification Σ_Y of f. Suppose σ is an i-dimensional stratum and assume it has been shown over all strata of dimension larger than i that $\mathscr{L}_f$ is prime. The local product structure discussed in Remark 2.6.(4) implies that the g in that remark inherits from f all hypotheses of the present theorem; compare Remark 2.8. We can therefore assume that σ is a point $\mathfrak{x}$ and calculate in $\mathscr{O}_{X,\mathfrak{x}}[y]$.

If $\mathscr{L}_f$ is not prime at the zero-dimensional stratum σ, it has a component primary to a prime ideal that contains the defining ideal of σ. Such component has dimension n or less in the cotangent bundle near σ. But by Theorem 3.16, $\mathscr{L}_f$ is a Cohen–Macaulay ideal of generic height $n-1$. So all primary components of $\mathscr{L}_f$ must have the same dimension, namely $\dim(\mathscr{L}_f) = n+1$. Hence $\mathscr{L}_f$ has no component over σ, and thus is a prime ideal at σ. $\square$

Remark 3.18. We have shown that, for Saito-holonomic and strongly Euler-homogeneous f, $\mathscr{L}_f$ is dimension $n+1$ and prime and Cohen–Macaulay over the tame locus. So for example, with these hypotheses, $\dim \mathscr{L}_f = n+1$ for all f with at most one-dimensional non-tame locus (*e.g.*, when $\dim X \le 5$).

Saito-holonomicity is necessary for this to be true. The divisor $f = xyz(x+y+z)(x+ay+bz)$ in $\mathbb{C}^5$ is tame, strongly Euler-homogeneous, and $C^\bullet_f$ is exact. Its logarithmic derivations vanish along the plane $x=y=z=0$ and thus $\mathscr{L}_f$ has a component of dimension 7 and is very much not a Cohen–Macaulay ideal of dimension 6. $\diamond$

Corollary 3.19. *Let $f \in \mathscr{O}_X$ be strongly Euler-homogeneous, Saito-holonomic, and tame. Then the ideal $\tilde{\mathscr{L}}_f$ of $\operatorname{gr}_{(0,1)}(\mathscr{D}_X[s])$ generated by the symbols of the operators of order one in $\operatorname{ann}_{\mathscr{D}_X[s]}(f^s)$ is a Cohen–Macaulay ideal of dimension n.*

Proof. All hypotheses and conclusions are local properties, so we may calculate in $\mathscr{O}_{X,\mathfrak{x}}[y]$, where f has Euler field $E_\mathfrak{x}$. The order one operators in $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[s]}(f^s)$ are $\mathbb{C}[s]\otimes_\mathbb{C}\operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f)\oplus\mathscr{O}_{X,\mathfrak{x}}[s]\cdot(E_\mathfrak{x}-s)$. Hence, $\tilde{\mathscr{L}}_{f,\mathfrak{x}}=\mathscr{L}_{f,\mathfrak{x}}+\mathscr{O}_X[y]\cdot(E_\mathfrak{x}\diamond y\,\mathrm{d}x)$. As $E_\mathfrak{x}\notin\operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f)$, we also have $(E_\mathfrak{x}\diamond y\,\mathrm{d}x)\notin\mathscr{L}_{f,\mathfrak{x}}$ as $\mathscr{L}_{f,\mathfrak{x}}$ is graded in y. By Theorem 3.17, $E_\mathfrak{x}\diamond y\,\mathrm{d}x$ is regular on $\mathscr{O}_{X,\mathfrak{x}}[y]/\mathscr{L}_{f,\mathfrak{x}}$. Hence $\mathscr{O}_{X,\mathfrak{x}}[y]/\tilde{\mathscr{L}}_{f,\mathfrak{x}}$ is a quotient of a Cohen–Macaulay ring by a regular element. □

Remark 3.20. At every $\mathfrak{x}\in X$ where $\Omega^1_X(\log f)$ is free, in the situation of Corollary 3.19, $\mathscr{L}_{f,\mathfrak{x}}$ and, *a fortiori*, $\tilde{\mathscr{L}}_{f,\mathfrak{x}}$ are local complete intersections. In particular, if f is free, satisfies the conditions of Corollary 3.19, and has a global homogeneity on affine X then $\mathscr{L}_f$ and $\tilde{\mathscr{L}}_f$ are complete intersections, compare [39]. For arrangements, related ideas in a different context have been worked out in [13] ⋄

Remark 3.21. Tameness might be a red herring in Corollary 3.19; we know of no case of a strongly Euler-homogeneous and Saito-holonomic divisor where $\tilde{\mathscr{L}}_f$ is not a Cohen–Macaulay ideal (even including the cases when $\mathscr{L}_f$ is not a Cohen–Macaulay ideal).

On the other hand, Saito-holonomicity is necessary: for the f from Remark 3.18, both $\mathscr{L}_f$ and $\tilde{\mathscr{L}}_f$ have dimension 7, but are generically (on X) of dimensions 6 and 5 respectively. ⋄

Remark 3.22. While the hypotheses of Theorem 3.16 apply to f if and only if they apply to f^k (they have similar logarithmic vector fields and Euler-homogeneities), it it not clear whether divisors with the same support are necessarily simultaneously free or tame. It would be interesting to understand the exponents over an arrangement that make it free (or tame); note that this question can be extended to complex exponents and then relates to multivariate Bernstein–Sato constructions, [5]. For example, the generic arrangement $xyz(x+y+z)$ is tame but not free, while $x^ay^bz^c(x+y+z)^d$ is "a free divisor" if $a+b+c+d=0$ in the sense that the relations between the generators of its partial derivatives form a free module. Similarly, the exponents $(-2,-2,-2,1,1,1,1,1,1)$ used in this sequence on the factors of the non-tame reduced arrangement in Example 5.7 give a tame "divisor" in this sense. ⋄

3.5. **Blowing up the Jacobian ideal.** The ring $\operatorname{gr}_{(0,1)}(\mathscr{D}_X)/\mathscr{L}_f$ is the symmetric algebra of the Jacobian ideal of f: $\mathscr{L}_f$ is the kernel of the map from $\operatorname{Sym}(\mathscr{O}_X^n)\longrightarrow\operatorname{Sym}(\operatorname{Jac}(f))$ induced by the presentation $(f_1,\ldots,f_n)$ of $\operatorname{Jac}(f)$. Symmetric algebras and their homological properties have been studied intensively for decades, since they form an approximation to the Rees algebra of an ideal. Indeed, $\mathscr{L}_f$ is the linear part (in our grading) of the kernel of the map that defines the blow-up of $\operatorname{Jac}(f)$.

We record here what we have shown about the symmetric algebra.

Corollary 3.23. *Assume that one of the following statements holds:*

- $n\le 3$*;*
- f *is strongly Euler-homogeneous, tame and Saito-holonomic.*

Then:

(1) *the regularity of the ideal* $\mathscr{L}_f$ *with respect to the cotangent variables* $y_1\ldots,y_n$ *is one: the Jacobian ideal is of linear type;*

(2) *the symmetric algebra and the Rees algebra of* $\operatorname{Jac}(f)$ *agree, and are Cohen–Macaulay domains.*

Proof. It suffices to consider the local case and we work over $S = \mathscr{O}_{X,\mathfrak{x}}[y]$. In both cases, $C_f^\bullet$ is a resolution of $S/\mathscr{L}_f$ and each C_f^i has a free resolution whose maps are independent of y. The first claim follows from the fact that the differential in $C_f^\bullet$ is linear in y. The second claim follows from Theorem 3.17 and 3.16 inasmuch as the symmetric algebra is concerned. If the Rees algebra were not to agree with the symmetric algebra, it would have to be of dimension less than $n+1$ under our hypotheses. But that is not possible, since its projectivization must be dimension n. □

Remark 3.24. (1) The previous result generalizes Theorem 5.6 in [9] where freeness and quasi-homogeneity was assumed.

(2) We do not know any instance where $C_f^\bullet$ is not a resolution, irrespective of *any* hypotheses. On the other hand, Torrelli pointed out that the ideal $\mathscr{O}_X \cdot f + \operatorname{Jac}(f)$ can be linear type at $\mathfrak{x}$ only if f is Euler-homogeneous at $\mathfrak{x}$, [8, Rmk. 1.26]

(3) For $f = xy(x+y)$, the ideal $\mathscr{L}_f$ is not normal. ⋄

3.6. **The annihilator of f^s.**

Lemma 3.25. *For $\mathfrak{x} \in \mathbb{C}^n$ let $f \in R = \mathscr{O}_{X,\mathfrak{x}}$, and let $\mathscr{P}$ be a set of differential operators in $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}}(f^s)$ such that the ideal sum in $R[y]$ of $\mathscr{L}_f$ with the ideal generated by $\operatorname{gr}_{(0,1)}(\mathscr{P})$ contains a prime ideal of height $n-1$. Then $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}}(f^s)$ is generated by $\operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f)$ and $\mathscr{P}$.*

Proof. By hypothesis, $\operatorname{gr}_{(0,1)}(\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}}(f^s))$ contains a prime ideal of height $n-1$. If $\operatorname{gr}_{(0,1)}(\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}}(f^s))$ were not equal to this prime ideal, it would be of height n or more. This contradicts what we know over a generic point of X, but all points have such points in every neighborhood. Hence $\operatorname{gr}_{(0,1)}(\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}}(f^s))$ is this prime ideal and generated by the symbols of $\mathscr{P}$ together with $\mathscr{L}_f$.

If $Q \in \operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}}(f^s)$, then for suitable $a_\delta, b_P \in \mathscr{D}_{X,\mathfrak{x}}$ (almost all of them zero) we have

$$\operatorname{gr}_{(0,1)}(Q) = \sum_{\delta \in \operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f)} \operatorname{gr}_{(0,1)}(a_\delta \cdot \delta) + \sum_{P \in \mathscr{P}} \operatorname{gr}_{(0,1)}(b_P \cdot P).$$

For graded (in y) ideals in $\mathscr{O}_{X,\mathfrak{x}}$, such as the ideal generated by $\mathscr{L}_{f,\mathfrak{x}}$ and the symbols of $\mathscr{P}$, this can be arranged in such a way that the orders of any $a_\delta \cdot \delta$ and $b_P \cdot P$ are equal to the order of Q. Then Q can be reduced modulo $\mathscr{D}_{X,\mathfrak{x}}(\mathscr{P}, \operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f))$ to an operator in $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}}(f^s)$ of lower order. By induction on this order, the lemma follows. □

Theorem 3.26. *If $f \in \mathscr{O}_X$ is tame, strongly Euler-homogeneous and Saito holonomic then $\operatorname{ann}_{\mathscr{D}_X[s]}(f^s)$ is generated at $\mathfrak{x} \in X$ by $\operatorname{Der}_X(\log_0(f))$ and any Euler-homogeneity $E_\mathfrak{x} - s$ where $E_\mathfrak{x} \bullet (f) = f$. If X is the analytic space to a smooth $\mathbb{C}$-scheme, this holds also in the algebraic category.*

Proof. Locally, this follows from Theorem 3.17 and Lemma 3.25. So the containment of sheaves $\mathscr{D}_X \cdot \operatorname{Der}_X(-\log_0 f) \subseteq \operatorname{ann}_{\mathscr{D}_X}(f^s)$ is an isomorphism. Now note that algebraic f has algebraic derivatives; all syzygies between the derivatives of f will then be algebraic. □

Example 3.27. Let $R = \mathbb{C}[x_{1,1}, \ldots, x_{m,n}]$ be a polynomial ring and let M be the matrix with $m_{i,j} = x_{i,j}$. It is well-known that for $n = m + 1$ the product f of the $m + 1$ maximal minors of M is a (linear) free divisor, see *e.g.* [27]. When $m = 3 = n - 1$, J. Martín-Morales informs us that (on the behest of L. Narvaez-Macarro) V. Levandovskyy and D. Andres have used the computer algebra system *Singular* [17] to determine that the annihilator of f^s is not cut out by operators of order one. Experimental evidence lets us believe that f is strongly Euler-homogeneous everywhere. Theorem 3.26 indicates that f should not be Saito-holonomic. And indeed, the locus where the logarithmic vector fields have rank less than 10 is 10-dimensional: it contains the variety of rank-deficient matrices. ⋄

4. Milnor fiber and Jacobian module

In this section, $n \geq 2$ and X is the algebraic variety $\mathbb{C}^n$. If $f \in \Gamma(X, \mathscr{O}_X) =: R_n$ denotes a homogeneous polynomial of degree d in n variables, then the Milnor fiber $M_{f,0}$ of f at the origin can be identified with the hypersurface $(f = 1)$.

If f has an isolated singularity at the origin $0 \in X$, Milnor established that the cohomology of $M_{f,0}$ is encoded in the residue ring of the Jacobian ideal: the Milnor fiber is a bouquet of $\mu = \dim_{\mathbb{C}}(\mathscr{O}_{X,0}/\operatorname{Jac}(f))$ many $(n-1)$-spheres.

Theorem 4.3 below can be seen as a partial extension to a class of more general homogeneous singularities. In these cases, the Jacobian ring is not necessarily Artinian, hence only parts of it can be responsible for cohomology classes on M_f. A relevant part of $R_n/\operatorname{Jac}(f)$ is selected by a local cohomology functor and we connect it to Hodge-theoretic data on M_f via logarithmic vector fields.

Remark 4.1. Let I be an ideal in a commutative ring R. We will make use of Grothendieck's *local cohomology functors* $H^\bullet_I(-)$, the right derived functors of the left-exact I-torsion functor $H^0_I(-) := \bigcup_t \operatorname{Hom}_R(R/I^t, -)$. For details we refer to [33]. ⋄

Notation 4.2. We use the term *simple normal crossing* to label a point in a divisor where locally a coordinate system exists in which the divisor takes the form $x_1^{a_1} \cdots x_n^{a_n}$, $a_i \in \mathbb{N}$, and where $(x_i = 0)$, $(x_j = 0)$ define globally distinct components of the divisor whenever $i \neq j$. The points of the variety $(g = 0)$ where g does not have simple normal crossings is called the *NSNC-locus of* g.

Theorem 4.3. *Let $f \in R_n$ be homogeneous of degree d, reduced, and suppose $n \geq 2$. Assume that $\operatorname{Proj}(R_n/f)$ has isolated singularities. Then, with $1 \leq k \leq d$ and $\lambda = \exp(2\pi\sqrt{-1}k/d)$,*

$$\dim_{\mathbb{C}}[H^0_{\mathfrak{m}}(R_n/\operatorname{Jac}(f))]_{d-n+k} \quad \leq \quad \dim_{\mathbb{C}} \operatorname{gr}^{\text{Hodge}}_{n-2}(H^{n-1}(M_{f,0}, \mathbb{C})_\lambda)$$

where the right hand side indicates the λ-eigenspace of the associated graded object to the Hodge filtration on $H^{n-1}(M_{f,0}, \mathbb{C})$.

Proof. We first give an outline of the proof of Theorem 4.3 and then fill in the details in five steps.

Let $S_n = \mathbb{C}[x_0, x_1, \ldots, x_n]$ be the coordinate ring of projective n-space. We usually denote x_0 by z and identify $\mathbb{P}^n \smallsetminus \operatorname{Var}(z)$ with $X = \mathbb{C}^n$. Then define a new polynomial $F \in S_n$ by $F(z, x_1, \ldots, x_n) = (f(x_1, \ldots, x_n) - z^d)z$. The corresponding homogeneous maximal ideals are denoted $\mathfrak{m} = (x_1, \ldots, x_n)R_n$ and $\mathfrak{n} = (x_0, \ldots, x_n)S_n$. The projective scheme defined by F is denoted $Y \subseteq \mathbb{P}^n$.

We show first that the singular locus of $\operatorname{Proj}(S_n/(f,z))$ is the same as the NSNC-locus of $\operatorname{Proj}(S_n/F)$ and so the hypotheses imply that the NSNC-locus of $\operatorname{Proj}(S_n/F)$ is zero-dimensional.

Next, choose an embedded resolution $\pi\colon (\mathbb{P},Y') \longrightarrow (\mathbb{P}^n,Y)$ of singularities that resolves only the NSNC-locus of $\operatorname{Proj}(S_n/F)$. As f is homogeneous, the resolution process takes place entirely inside the (preimage of the) locus defined by $\operatorname{Var}(z)$. Since the Milnor fiber $M_{f,0}$ is $\operatorname{Var}(f-1)\subseteq X$, it agrees with $\operatorname{Var}(f-z^d)\smallsetminus\operatorname{Var}(z)$ inside $\mathbb{P}^n_{\mathbb{C}}$. In the resolved model, the strict transform of $f-z^d$ is smooth, and so the cohomology of $M_{f,0}$ is determined by the cohomology of logarithmic vector fields on the strict transform of $\operatorname{Var}(f-z^d)$ along (the preimage of the reduced divisor defined by) $\operatorname{Var}(z)$. Long exact sequences can be used to translate the issue to cohomology of logarithmic vector fields on $\mathbb{P}$ along (the reduced divisor defined by) $\operatorname{Var}(F)$. If the NSNC-locus of $\operatorname{Proj}(S_n/F)$ is zero-dimensional, it can be shown that the appropriate cohomology of logarithmic vector fields along $\operatorname{Var}(\pi^*(F))$ and along its reduced divisor are closely related: the Leray spectral sequence allows to connect both to cohomology of logarithmic vector fields on $\mathbb{P}^n$ along $\operatorname{Var}(F)$. We then use local cohomology methods to relate such cohomology to torsion in the Jacobian ring of F and of f.

Step 0: *The NSNC-locus of* $\operatorname{Proj}(S_n/F)$.

We have $\operatorname{Jac}(F) = (z\operatorname{Jac}(f),(d+1)z^d-f)S_n$ so that the singular locus of $\operatorname{Proj}(S_n/F)$ is defined by $(z=f=0)$. In particular, the singular (and hence the NSNC-) locus of F are inside $(z=0)$.

Without loss of generality we take the chart $(x_n\neq 0)$ of $\operatorname{Proj}(S_n)$ and show that in this chart all points outside $(z,\operatorname{Jac}(f))S_n$ are simple normal crossing points of F. In terms of coordinates on this chart, write $y_i=x_i/x_n$ and $t=z/x_n$, and write $g(y_1,\ldots,y_{n-1})$ for $f(x_1,\ldots,x_n)/x_n^d$.

Imagine a point p outside $\operatorname{Var}(z,\operatorname{Jac}(f))$ that fails to have simple normal crossings. Since p must be a singular point of F, we will have $z=f=0$ at p and so p is in $\operatorname{Var}(z)$ but (therefore) not in $\operatorname{Var}(\operatorname{Jac}(f))$. So, some derivative f_ℓ does not vanish at p and so $f-z^d$ is smooth in p.

In order for p to be in the NSNC-locus of F it is necessary that the gradients of t and of $g-t^d$ be linearly dependent at p, which implies that the gradient of g must be zero in p. As $g(y_1,\ldots,y_{n-1},1)=f(x_1,\ldots,x_n)/x_n^d$ the chain rule implies that $\frac{\partial(f(x_1,\ldots,x_n)/x_n^d)}{\partial x_i} = \sum_{j=1}^{n-1}\frac{\partial g}{\partial y_j}\cdot\frac{\partial y_j}{\partial x_i}$. If $\ell\neq n$ this states that $\frac{\partial f}{\partial x_\ell}(y_1,\ldots,y_{n-1},1) = \frac{\partial g}{\partial y_\ell}$ while for $\ell=n$ it states that $(-d)x_n^{-d}f(x_1,\ldots,x_n)+x_n^{-d+1}f_n(x_1,\ldots,x_n) = -\sum_{j=1}^{n-1}y_j\cdot g_j(y_1,\ldots,y_{n-1})$.

As the gradient of g vanishes at p, so do all derivatives of g. By the chain rule above, each f_j, $1\le j\le n-1$, is zero in p. By the chain rule for the x_n-derivative above, since f vanishes at p but x_n does not, we conclude that f_n is zero at p as well. In other words, p is in $\operatorname{Var}(\operatorname{Jac}(f),z)$. By contradiction, the NSNC-locus of F is contained in $\operatorname{Var}(z,\operatorname{Jac}(f))$.

Conversely, take $p\in\operatorname{Var}(z,\operatorname{Jac}(f))\subseteq\operatorname{Var}(z,z^d-f)$ on the chart $x_n\neq 0$. Then the gradient of g is zero at p and so p is an NSNC-point of F as $(g=t^d)$ and $(t=0)$ do not meet normally at p.

Step 1: *From* $R_n/\operatorname{Jac}(f)$ *to* $\Omega^{n-1}_{\mathbb{P}^n}(\log F)$.

As R_n-module, $S_n/\operatorname{Jac}(F)$ is generated by the cosets of $z^0,\dots,z^{d-1}$, and these are R_n-independent:

$$S_n/\operatorname{Jac}(F) \cong \bigoplus_{i=0}^{d-1} R_n \cdot (z^i \bmod \operatorname{Jac}(F)).$$

Observe that the R_n-annihilator of $(z^i \bmod \operatorname{Jac}(F))$ is $\operatorname{Jac}(f)$ if $1 \le i \le d-1$, and that the R_n-annihilator of $(1 \bmod \operatorname{Jac}(F))$ is $f \cdot \operatorname{Jac}(f)$. Since z is nilpotent on $S_n/\operatorname{Jac}(F)$, the concepts of $\mathfrak{m}$-torsion and $\mathfrak{n}$-torsion agree on this module. Then, as graded R_n-modules,

$$H := H^0_{\mathfrak{n}}(S_n/\operatorname{Jac}(F)) = \bigoplus_{i=1}^{d-1} H^0_{\mathfrak{m}}(R_n/\operatorname{Jac}(f))(-i) \oplus H^0_{\mathfrak{m}}(R_n/\operatorname{Jac}(f))(-d)$$

since $H^0_{\mathfrak{m}}(R_n/f\cdot\operatorname{Jac}(f)) = H^0_{\mathfrak{m}}(f\cdot R_n/f\cdot\operatorname{Jac}(f)) = H^0_{\mathfrak{m}}(R_n/\operatorname{Jac}(f))(-d)$.

Now let $Z \subseteq \bigoplus_{i=0}^n S_n \cdot e_i$ be the syzygy module on the partial derivatives $F_z = F_0, F_1, \dots, F_n$ where $F_i = \partial F/\partial x_i$. It inherits a natural grading via $\deg(a_i e_i) = \deg(a_i) + d$, since $\deg(F_i) = d$. The start of the minimal S_n-graded resolution of $S_n/\operatorname{Jac}(F)$ is hence

$$0 \longrightarrow Z \longrightarrow S_n(-d)^{n+1} \longrightarrow S_n \longrightarrow S_n/\operatorname{Jac}(F) \longrightarrow 0.$$

It follows from the long exact sequence of local cohomology that, as graded modules, $H = H^0_{\mathfrak{n}}(S_n/\operatorname{Jac}(F)) \cong H^1_{\mathfrak{n}}(\operatorname{Jac}(F)) \cong H^2_{\mathfrak{n}}(Z)$.

The module $\operatorname{Der}_{S_n}(-\log_0 F)$ of derivations annihilating F inherits a natural grading via $\deg(\sum_{i=0}^n a_i\partial_i) = \deg(a_i) - 1$. As graded modules, $\operatorname{Der}_{S_n}(-\log_0 F) \cong Z(d+1)$. Thus,

$$H^2_{\mathfrak{n}}(Z) \cong H^2_{\mathfrak{n}}(\operatorname{Der}_{S_n}(-\log_0 F))(-d-1).$$

Since F is homogeneous, the Euler derivation $E = \sum_{i=0}^n x_i\partial_i$ is in $\operatorname{Der}_{S_n}(-\log F)$, and there is a splitting $S_n \cdot E \oplus \operatorname{Der}_{S_n}(-\log_0 F) \cong \operatorname{Der}_{S_n}(-\log F)$. Since $S_n \cdot E$ is free and $n \ge 2$, the splitting gives

$$H^2_{\mathfrak{n}}(\operatorname{Der}_{S_n}(-\log_0 F))(-d-1) \cong H^2_{\mathfrak{n}}(\operatorname{Der}_{S_n}(-\log F))(-d-1).$$

The differential forms $\Omega^\bullet_{S_n}$ on $\mathbb{C}^{n+1}$ form a graded algebra with differential $\mathrm{d}\colon \Omega^0_{S_n} = S_n \longrightarrow \Omega^1_{S_n}$ homogeneous of degree zero. The modules $\Omega^\bullet_{S_n}(\log F)$ inherit a natural grading from being a submodule of $\frac{1}{F}\cdot\Omega^\bullet_{S_n} \cong \Omega^\bullet_{S_n}(d+1)$, and as graded modules $\operatorname{Der}_{S_n}(-\log F) \cong \Omega^n_{S_n}(\log F)(n-d)$, cf. Remark 3.4.(5).

The module $\Omega^{n-1}_{S_n}(\log_E F) \cong \Omega^n_{S_n}(\log_0 F)$ is the contraction of $\Omega^n_{S_n}(\log F)$ with E; this surjection is dual to the inclusion $\operatorname{Der}_{S_n}(-\log_0 F) \hookrightarrow \operatorname{Der}_{S_n}(-\log F)$ under the above identification. Hence there is an induced graded isomorphism

$$\operatorname{Der}_{S_n}(-\log_0 F) \cong \Omega^{n-1}_{S_n}(\log_E F)(n-d)$$

sending $\delta \mapsto \delta \diamond (E \diamond (\mathrm{d}x/F))$, compare [22].

Pasting together all previous identifications,

$$\begin{aligned} H^2_{\mathfrak{n}}(\Omega^{n-1}_{S_n}(\log_E F)) &\cong H^2_{\mathfrak{n}}(\operatorname{Der}_{S_n}(-\log_0 F))(d-n) \\ &\cong H^2_{\mathfrak{n}}(Z)(2d-n+1) \\ &\cong H(2d-n+1) \\ &\cong \bigoplus_{i=1}^{d} H^0_{\mathfrak{m}}(R_n/\operatorname{Jac}(f))(d-n+i). \end{aligned}$$

Let $\tilde\Omega^{n-1}_{S_n}(\log_E F)$ be the sheaf on $\mathbb{P}^n$ induced by $\Omega^{n-1}_{S_n}(\log_E F)$. By [22, Proposition 2.11] (valid for general homogeneous divisors), $\tilde\Omega^{n-1}_{S_n}(\log_E F)$ is the sheaf $\Omega^{n-1}_{\mathbb{P}^n}(\log F)$ of $(n-1)$-forms on $\mathbb{P}^n$ that are logarithmic in the sense of K. Saito along $Y = \operatorname{Proj}(S/F)$ on all charts. The Grothendieck–Serre correspondence [3, Thm. 20.4.4] yields then a graded S_n-module isomorphism

$$H^2_{\mathfrak{n}}(\Omega^{n-1}_{S_n}(\log_E F)) \cong \bigoplus_{t\in\mathbb{Z}} H^1(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(\log F)(t)).$$

In particular,

$$\begin{aligned}\bigoplus_{i=1}^{d}\left[H^0_{\mathfrak{m}}(R/\operatorname{Jac}(f))\right]_{d-n+i} &= \left[\bigoplus_{i=1}^{d} H^0_{\mathfrak{m}}(R/\operatorname{Jac}(f))(d-n+i)\right]_0\\ &= \left[H^2_{\mathfrak{n}}(\Omega^{n-1}_{S_n}\log_E F)\right]_0 = H^1(\mathbb{P}^n, \tilde\Omega^{n-1}_{S_n}(\log_E F))\\ &= H^1(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(\log F)).\end{aligned}$$

Step 2: *The Leray spectral sequence.*

Let $\pi\colon (\mathbb{P}, Y') \longrightarrow (\mathbb{P}^n, Y)$ be an embedded resolution of singularities, with $\mathbb{P}$ smooth and $F' := \pi^*(F)$ having simple normal crossings. Since the singularities of F are inside $\operatorname{Var}(z)$, one can arrange that $\mathbb{P}$ and $\mathbb{P}^n$ agree where $z \neq 0$. Indeed, functoriality of the resolution process implies that one only needs to resolve the NSNC-locus of $\operatorname{Proj}(S_n/F)$.

Note that while F is reduced, this is not so for F'. Let F'_{red} define the reduced divisor. Deligne's logarithmic sheaves $\Omega^i_{\mathbb{P}}(Y')$ consider reduced underlying schemes and are our sheaves $\Omega^i_{\mathbb{P}}(\log(F'_{\text{red}}))$, usually properly contained in $\Omega^i_{\mathbb{P}}(\log(F'))$.

The cokernel Q of the inclusion $\pi_*(\Omega^{n-1}_{\mathbb{P}}(\log F'_{\text{red}})) \subseteq \pi_*(\Omega^{n-1}_{\mathbb{P}}(\log F'))$ is supported in the NSNC-locus of F. If this locus is of dimension zero, $H^{>0}(\mathbb{P}^n, Q) = 0$ and so there is a natural surjection

$$H^1(\mathbb{P}^n, \pi_*(\Omega^{n-1}_{\mathbb{P}}(\log F'_{\text{red}}))) \twoheadrightarrow H^1(\mathbb{P}^n, \pi_*(\Omega^{n-1}_{\mathbb{P}}(\log F'))) \tag{4.1}$$

(and the higher cohomology groups agree).

On the other hand, the Leray spectral sequence induces a natural embedding

$$H^1(\mathbb{P}^n, \pi_*(\Omega^{n-1}_{\mathbb{P}}(\log F')) \hookrightarrow H^1(\mathbb{P}, \Omega^{n-1}_{\mathbb{P}}(\log F')).$$

By Lemma 6.2, $\pi_*(\Omega^{n-1}_{\mathbb{P}}(\log F')) = \Omega^{n-1}_{\mathbb{P}^n}(\log F)$ and so we have a natural diagram

$$\begin{array}{ccc} H^1(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(\log F)) = H^1(\mathbb{P}^n, \pi_*(\Omega^{n-1}_{\mathbb{P}}(\log F'))) & \hookrightarrow & H^1(\mathbb{P}, \Omega^{n-1}_{\mathbb{P}}(\log F')) \\ & & \uparrow\!\!\uparrow \\ & & H^1(\mathbb{P}, \Omega^{n-1}_{\mathbb{P}}(\log F'_{\text{red}})) \end{array}$$

as long as $H^1(\mathbb{P}^n, Q) = 0$. In particular, the dimension of $H^1(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(\log F))$ is bounded above by the dimension of $H^1(\mathbb{P}, \Omega^{n-1}_{\mathbb{P}}(\log F'_{\text{red}})) = H^1(\mathbb{P}, \Omega^{n-1}_{\mathbb{P}}(Y'))$.

Step 3: *Matching with Deligne's results.*

Consider for the moment a smooth projective variety W with a reduced simple normal crossing divisor $V = \bigcup V_i$ and a distinguished component V_0. Then, by [25, Properties 2.3] there is a short exact sequence of sheaves

$$0 \longrightarrow \Omega^i_W(\log(V - V_0)) \longrightarrow \Omega^i_W(\log(V)) \longrightarrow \Omega^{i-1}_{V_0}(\log(V - V_0)|_{V_0}) \longrightarrow 0.$$

With $i = n-1$, it induces a long exact sequence

(4.2)

$$\ldots \longrightarrow H^1(W, \Omega_W^{n-1}(\log(V - V_0))) \longrightarrow H^1(W, \Omega_W^{n-1}(\log(V))) \longrightarrow$$
$$\longrightarrow H^1(V_0, \Omega_{V_0}^{n-2}(\log(V - V_0)|_{V_0})) \longrightarrow H^2(W, \Omega_W^{n-1}(\log(V - V_0))) \longrightarrow \ldots$$

Deligne's theory [18, 19] implies that

$$\begin{aligned} H^i(W, \Omega_W^j(\log(V - V_0))) &\cong \operatorname{gr}_j^{\text{Hodge}} H_{\mathrm{d}R}^{i+j}(W \smallsetminus (V - V_0)); \\ H^i(W, \Omega_W^j(\log V)) &\cong \operatorname{gr}_j^{\text{Hodge}} H_{\mathrm{d}R}^{i+j}(W \smallsetminus V); \\ H^i(V_0, \Omega_{V_0}^j(\log(V - V_0)|_{V_0})) &\cong \operatorname{gr}_j^{\text{Hodge}} H_{\mathrm{d}R}^{i+j}(V_0 \smallsetminus (V - V_0)). \end{aligned}$$

Here, $V - V_0$ is the difference of divisors, $W \smallsetminus V$ the set-theoretic difference.

The situation we are interested in is when $W = \mathbb{P}$, $V = Y'$, and V_0 is the strict transform of $\operatorname{Div}(f - z^d)$. In that case, $W \smallsetminus (V - V_0)$ is affine space $\mathbb{C}^n = \mathbb{P}^n \smallsetminus \operatorname{Var}(z)$ while $V_0 \smallsetminus (V - V_0)$ is the smoothly compactified Milnor fiber minus the part at infinity, hence exactly $M_{f,0}$. In the 4-term exact sequence (4.2), the left- and right-most terms are zero as $\mathbb{C}^n$ is contractible; we obtain a monomorphism $H^1(\mathbb{P}, \Omega_{\mathbb{P}}^{n-1}(\log F'_{\text{red}})) \hookrightarrow \operatorname{gr}_{n-2}^{\text{Hodge}} H^{n-1}(M_{f,0}, \mathbb{C})$.

Collecting the results from each step, one obtains

$$\bigoplus_{i=1}^{d} \left[H_{\mathfrak{m}}^0(R_n/\operatorname{Jac}(f))\right]_{d-n+i} = H^1(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{n-1}(\log F)) \hookrightarrow H^1(\mathbb{P}, \Omega_{\mathbb{P}}^{n-1}(\log F'))$$
$$\uparrow$$
$$H^1(\mathbb{P}, \Omega_{\mathbb{P}}^{n-1}(\log F'_{\text{red}}))$$
$$\downarrow$$
$$\operatorname{gr}_{n-2}^{\text{Hodge}}(H^{n-1}(M_{f,0}, \mathbb{C})).$$

Step 4: *Monodromy.*

The monodromy isomorphism on $H^i(M_{f,0})$ is induced by the geometric monodromy $x_i \longrightarrow \omega_d x_i$ where ω_d is a primitive d-th root of unity. One extends it to a graded automorphism of S_n that fixes $\operatorname{Var}(F)$. Thus, the grading and the geometric monodromy extend to the pair $(\mathbb{P}^n, Y)$ and functoriality of resolution of singularities [51] guarantees an extension to π. In particular, all morphisms in the above display are graded and monodromy-equivariant (apart from the combined effect of identification of logarithmic derivations with logarithmic differentials and the residue map, corresponding to a twist by $(\omega_d)^n$). Since the graded components of the module $H_{\mathfrak{n}}^0(S_n/\operatorname{Jac}(F))$ are ω_d-eigenspaces, the claim follows. □

Remark 4.4. (1) If $n = 3$, the condition on the singular locus is automatic.

(2) The current form of Theorem 4.3 evolved from a previous one through helpful comments and criticisms by M. Saito. He has produced a similar proof that is based on cyclic covers rather than the closure of the Milnor fiber in $\mathbb{P}^n$, see the appendix of [44].

(3) If the natural map (4.1) is an isomorphism, one obtains a natural map $[H_{\mathfrak{m}}^0(R_n/\operatorname{Jac}(f))]_{d-n+i} \hookrightarrow \operatorname{gr}_{n-2}^{\text{Hodge}}(H^{n-1}(M_f, \mathbb{C})_\lambda)$. There seems to be no natural lift to $H^{n-1}(M_f, \mathbb{C})_\lambda$, as was pointed out by M. Saito.

(4) Nonzero elements in $H^{n-1}(M_f, \mathbb{C})$ can be related to roots of $b_f(s)$ under certain circumstances, see for example [42, 45, 49]. M. Saito has produced examples that show that there is no obvious relation between $b_f(s)$ and $H^0_{\mathfrak{n}}(S_n/\operatorname{Jac}(F))$, even if (4.1) is an isomorphism, see [44]. $\diamond$

5. Hyperplane arrangements

In this section we consider hyperplane arrangements $\mathscr{A} \subseteq \mathbb{C}^n =: X$ defined by

$$f_{\mathscr{A}} = \prod_1^d L_i \in R_n := \mathbb{C}[x],$$

where the L_i are (not necessarily homogeneous and not necessarily distinct) polynomials of degree one. Arrangements form an interesting class of strongly Euler-homogeneous and Saito-holonomic divisors. The canonical Whitney stratification, the logarithmic stratification, and the stratification by multiplicity all agree for arrangements.

Let D_n be the n-th Weyl algebra $R_n\langle\partial\rangle$ and set as before

$$E = \sum_{i=1}^n x_i \partial_i \in D_n = R_n\langle\partial\rangle.$$

5.1. **The differential annihilator.** Around the time [46] was published, Terao conjectured that $\operatorname{ann}_{D_n}(1/f_{\mathscr{A}})$ be generated by operators of order one for every reduced hyperplane arrangement. In [12] it is shown that for locally weakly quasi-homogeneous free polynomials f (which includes free arrangements), $\operatorname{ann}_{D_n}(1/f^k)$ is generated by operators of order one for $k \gg 0$, compare also [39, Rem. 1.7.4]. Terao's conjecture was affirmed for generic arrangements by Torrelli, who also raised the corresponding question about the generic annihilator $\operatorname{ann}_{D_n}(f^s_{\mathscr{A}})$ [47]. Related results are contained in [32].

We prove here that Terao's original conjecture is correct. Moreover, we confirm the generic annihilator conjecture in a large class of examples, but disprove it in general.

Definition 5.1. A *central* arrangement is an arrangement defined by a polynomial $f_{\mathscr{A}}$ that is homogeneous in the standard sense.

The following is a direct consequence of Theorem 3.26:

Theorem 5.2. *If the arrangement $\mathscr{A} \subseteq \mathbb{C}^n$ is tame then $\operatorname{ann}_{D_n}(f^s_{\mathscr{A}})$ is generated by operators of order one.* $\square$

This result implies immediately the validity of Terao's conjecture for tame arrangements. Indeed, we can assume that $f_{\mathscr{A}}$ is central of degree d, with a global Euler-homogeneity E. Then $\operatorname{ann}_{D_n[s]}(f^s_{\mathscr{A}})$ is (globally) generated by operators of order one, namely by $E - ds$ and $\operatorname{Der}_{\mathbb{C}^n}(-\log_0 \mathscr{A})$. The only integral root of the Bernstein–Sato polynomial of an arrangement is -1 by [49]. Consider the evaluation morphism $\operatorname{ann}_{D_n[s]}(f^s_{\mathscr{A}}) \longrightarrow \operatorname{ann}_{D_n}(1/f_{\mathscr{A}})$ that sends s to -1. By [35, Prop. 6.2], this morphism is surjective, hence the target is generated by order one operators. We show next that tameness is not required: Terao's conjecture holds in fact in full generality.

Theorem 5.3. *The differential annihilator $\operatorname{ann}_D(1/f_{\mathscr{A}})$ is generated by operators of order one.*

Proof. If the rank r of the arrangement is less than n then a suitable change of coordinates brings $f_{\mathscr{A}}$ into the ring $\mathbb{C}[x_1,\ldots,x_r]$. The truth of the statement of the theorem then only depends on the corresponding arrangement in $\mathbb{C}^r$. Without loss of generality we hence may (and will) assume that $r=n$.

Before getting to the main part of the proof we need to set up some notation. Denote for a subset I of $\{1,\ldots,d\}$ by L_I the factor $\prod_{i\in I} L_i$ of $f_{\mathscr{A}}$. Let

$$\mathscr{L}^\bullet\colon 0 \longrightarrow \underbrace{R_n}_{\text{degree }0} \longrightarrow \underbrace{\bigoplus R_n[1/L_i]}_{\text{degree }1} \longrightarrow \cdots \longrightarrow \underbrace{\bigoplus_{|I|=d-1} R_n[1/L_I]}_{\text{degree }d-1} \longrightarrow \underbrace{R_n[1/f_{\mathscr{A}}]}_{\text{degree }d} \longrightarrow 0$$

be the Čech complex attached to the family $\{L_1,\ldots,L_d\}$. Since the rank of $\mathscr{A}$ is n, the ideal generated by $L_1,\ldots,L_d$ is the maximal graded ideal $\mathfrak{m}=R_n\cdot x$ where $x=(x_1,\ldots,x_n)$. By standard facts of local cohomology (see [33]), the complex $\mathscr{L}^\bullet$ has exactly one cohomology group, positioned in cohomological degree n.

The Čech complex is a complex of D_n-modules since its constituents are localizations of the tautological D_n-module R_n. As such, its unique cohomology group is the direct image of the structure sheaf of the origin under the map that embeds the origin in $\mathbb{C}^n$. In particular, $H^n(\mathscr{L}^\bullet)$ is isomorphic to $D_n/D_n\cdot x$.

For $I\subseteq\{1,\ldots,n\}$, let η_I be the element in $\mathscr{L}^{|I|}$ whose component in $R_n[1/L_I]$ is $1/L_I$, and which is zero in all other components. The coboundary map of $\mathscr{L}^\bullet$ is just the signed sum of all inclusions $R_n[1/L_I]\hookrightarrow R_n[1/L_{I\cup\{j\}}]$. It thus sends η_I to $\sum(-1)^{\operatorname{sgn}(I,j)}L_j\cdot\eta_{I\cup\{j\}}$ where the sum runs over elements j in $\{1,\ldots,d\}\smallsetminus I$ and where $\operatorname{sgn}(I,j)$ is -1 raised to the number of elements of I that are greater than j.

Now let $\mathscr{D}^\bullet$ be the Koszul cocomplex associated to right-multiplication by the elements $L_1,\ldots,L_d$ on D_n. The module $\mathscr{D}^k$ is the direct sum of copies $D_n\cdot e_I$ of D_n where I runs over the subsets of $\{1,\ldots,d\}$ of size k and e_I is simply a symbol. With the right choices, the coboundary map in $\mathscr{D}^\bullet$ sends $1\cdot e_I$ in $\mathscr{D}^k$ to $\sum(-1)^{\operatorname{sgn}(I,j)}L_j\cdot e_{I\cup\{j\}}$ in $\mathscr{D}^{k+1}$.

There is a morphism of complexes $\phi\colon\mathscr{D}^\bullet\longrightarrow\mathscr{L}^\bullet$ that sends e_I to η_I. Since -1 is the only integral root of the Bernstein–Sato polynomial of an arrangement, η_I generates $R_n[1/L_I]$ over D_n and so ϕ is surjective. For each index set I denote by K_I the kernel of the map $\phi_I\colon D_n\cdot e_I\longrightarrow D_n\bullet\eta_I=R_n[1/L_I]$. Then there are induced maps $K_I\longrightarrow K_{I\cup\{j\}}$ via right-multiplication by L_j for each $j\in\{1,\ldots,d\}\smallsetminus I$ and thus an induced short exact sequence

$$0\longrightarrow\mathscr{K}^\bullet\longrightarrow\mathscr{D}^\bullet\longrightarrow\mathscr{L}^\bullet\longrightarrow 0$$

of complexes, where K_I becomes the I-th constituent of $\mathscr{K}^t=\bigoplus_{|I|=t}K_I$.

We now start the actual proof of the theorem. Since the rank of our arrangement is n, for $d=n$ we are looking at a Boolean arrangement and in that case the theorem is easy to check. Assume now that $d>n$. The long exact sequence of cohomology attached to the above short exact sequence of complexes ends with

$$H^{d-1}(\mathscr{D}^\bullet)\xrightarrow{\alpha}H^{d-1}(\mathscr{L}^\bullet)\longrightarrow H^d(\mathscr{K}^\bullet)\xrightarrow{\beta}\underbrace{H^d(\mathscr{D}^\bullet)}_{=D_n/D_n\cdot x}\longrightarrow H^d(\mathscr{L}^\bullet)=0, \tag{5.1}$$

compare the second paragraph of the proof.

Our first contention regarding this sequence is that α is surjective. When $d-1>n$ then this is automatic as then the target module vanishes. In the case $d-1=n$ denote $L_{\hat r}$ the polynomial $f_{\mathscr{A}}/L_r$ and observe that the kernel of $\mathscr{L}^{d-1}\longrightarrow\mathscr{L}^d$

consists of the cochains $(g_1/L^k_{\hat 1}, \dots, g_d/L^k_{\hat d}) \in \mathscr{L}^{d-1}$ for which $\sum(-1)^j g_j L^k_j = 0$. Since $d-1 = n$ equals the rank of the arrangement, there is an (up to scaling) unique $\mathbb{C}$-linear syzygy $\sum(-1)^r c_r L_r = 0$, $c_r \in \mathbb{C}$, between $L_1, \dots, L_d$. The corresponding element $c_L := \oplus c_r/L_{\hat r} \in \mathscr{L}^{d-1}$ is a cochain and at least one of the polynomials $L_{\hat r}$ with $c_r \neq 0$ is the product of n linearly independent linear forms. We briefly consider the Čech complex on the forms $L_1, \dots, L_{r-1}, L_{r+1}, \dots, L_d$ generating the ideal $\mathfrak{m}$. The n-th cohomology group of this complex is $H^n_{\mathfrak{m}}(R_n) = D_n/D_n \cdot x$ and generated over D_n by $c_r/L_{\hat r}$. It follows that $c_r/L_{\hat r}$ is not a sum of fractions where each denominator is a proper factor of $L_{\hat r}$. Returning to $\mathscr{L}^\bullet$, the cochain $c_L = \oplus c_r/L_{\hat r} \in \mathscr{L}^{d-1}$ cannot be in the image of $\mathscr{L}^{d-2} \longrightarrow \mathscr{L}^{d-1}$ as follows by looking at the $\hat r$-component. We have thus identified a nonzero element c_L in $H^{d-1}(\mathscr{L}^\bullet)$; since this module is isomorphic to $D_n/D_n \cdot x$ it is a simple D_n-module and thus c_L generates it. The claim on the surjectivity of α then follows since the Koszul cochain $\oplus c_i \cdot e_{\hat r} \in \mathscr{D}^{d-1}$ maps to c_L under α and to 0 under the coboundary.

We now return to our initial assumption $d > n$ but otherwise arbitrary, and note that, since α is surjective, the exact sequence (5.1) simplifies to a short exact sequence

$$0 \longrightarrow \operatorname{im}(\mathscr{K}^{d-1} \longrightarrow \mathscr{K}^d) \hookrightarrow \mathscr{K}^d \longrightarrow D_n/D_n \cdot x \longrightarrow 0.$$

Since the map $(\mathscr{K}^{d-1} \longrightarrow \mathscr{K}^d)$ is induced from $(\mathscr{D}^{d-1} \longrightarrow \mathscr{D}^d)$, it follows that $\operatorname{im}(\mathscr{K}^{d-1} \longrightarrow \mathscr{K}^d)$ is simply the ideal sum $\sum K_{L_{\hat r}} \cdot L_r$ in D_n. By induction, the ideals $K_{L_{\hat r}} \cdot L_r \subseteq D_n \cdot x$ are generated by operators of order one. Since the cokernel $D_n/D_n \cdot x$ of the inclusion is simple, the proof of the theorem will be complete if we can exhibit an order one differential operator in $\mathscr{K}^d$ that is not in $D_n \cdot x$. Such operator is given by the Euler operator $\sum x_r \partial_r + d = \sum \partial_r x_r - n + d$ since we are assuming $d > n$ and so in particular $-n + d \neq 0$. □

5.2. **Logarithmic b-functions.** Local systems on the open complement are an important tool for the study of arrangements, as they connect to cohomology of Milnor fibers (and other covers) and resonance varieties. One way to view a local system is as a D_n-module of the form $D_n \cdot f^\alpha$, $\alpha \in \mathbb{C}$. It is therefore of interest to know the structure of such modules.

For all locally quasi-homogeneous free divisors $f \in R_n$, the D_n-annihilator of the section $1/f$ of the D_n-module $R_n[1/f]$ is at any point generated by derivations, see [11]. The question how this compares to $\operatorname{ann}_{D_n}(f^s)$ being generated by order one operators is an interesting one. A related property is that $b_f(s)$ have no negative integer roots outside $s = -1$, as is the case for hyperplane arrangements, [49, Thm. 5.1]. It turns out, this condition is not sufficient, even for arrangements, see Example 5.7 below. Nonetheless, the derivations are never too far from the annihilator for an arbitrary arrangement as we show now.

Lemma 5.4. *For any arrangement $\mathscr{A}$ there is a finite list Q of rational numbers such that if $\alpha - \mathbb{N}$ is disjoint to Q then $\operatorname{ann}_{D_n}(f^\alpha_{\mathscr{A}})$ is generated by derivations.*

Proof. We shall show inductively the following claim, which implies the lemma. We denote $\operatorname{ann}^{(1)}_{D_n[s]}(f^s_{\mathscr{A}})$ the ideal generated by the operators of order one in $\operatorname{ann}_{D_n[s]}(f^s_{\mathscr{A}})$.

Claim. For each logarithmic stratum σ of $\operatorname{Var}(f_{\mathscr{A}})$ there is a polynomial $b^{\sigma}_{f_{\mathscr{A}}}(s)$ such that

$$\left(\prod_{\tau\supseteq\sigma} b^{\tau}_{f_{\mathscr{A}}}(s)\right)\cdot \operatorname{ann}_{D_n[s]}(f^s_{\mathscr{A}}) \subseteq \operatorname{ann}^{(1)}_{D_n[s]}(f^s_{\mathscr{A}}) \text{ along } \sigma.$$

Granting this claim, the lemma follows with Q as the root set of $b_{f_{\mathscr{A}}}(s)\cdot \operatorname{lcm}_\sigma\left(b^{\sigma}_{f_{\mathscr{A}}}(s)\right)$. Indeed, with α not a root of this product, $\operatorname{ann}_{D_n}(f^s_{\mathscr{A}})$ evaluated at $s=\alpha$ is inside $D_n\cdot\operatorname{ann}^{(1)}_{D_n}(f^\alpha)$. By [35, Prop. 6.2], if $\alpha-\mathbb{N}$ is disjoint to the root set of $b_{f_{\mathscr{A}}}(s)$ then evaluating $\operatorname{ann}_{D_n[s]}(f^s_{\mathscr{A}}) = D_n[s](\operatorname{ann}_{D_n}(f^s_{\mathscr{A}}), E-ds)$ at $s=\alpha$ gives all of $\operatorname{ann}_{D_n}(f^\alpha_{\mathscr{A}})$ which is thus generated by derivations.

The Claim is clear in the tame case. By induction we can assume that the claim holds along all positive-dimensional strata of the logarithmic stratification of the arrangement. (During induction, along σ, we may replace $f_{\mathscr{A}}$ by $\prod_{L_i(\sigma)=0} L_i$; this changes the defining equation only by a unit). Let $\tilde{b}(s)$ be the least common multiple of the polynomials $b^{\sigma}_{f_{\mathscr{A}}}(s)$ where σ is positive-dimensional. Then choose a zero-dimensional stratum o, identify it with the origin of $\mathbb{C}^n$, and let $E=\sum_{i=1}^n x_i\partial_i$ denote the corresponding Euler vector field.

Arguing locally, we can now assume that $f_{\mathscr{A}}$ is central of degree d and that $\tilde{b}(s)\frac{\operatorname{ann}_{D_n[s]}(f^s_{\mathscr{A}})}{\operatorname{ann}^{(1)}_{D_n[s]}(f^s_{\mathscr{A}})}$ is supported only at the origin. Since s acts through the Euler operator, it follows that this module is D_n-coherent and hence holonomic. Let $\overline{P}$ be in the socle Σ of this quotient module; since the generators of $\operatorname{ann}_{D_n[s]}(f^s_{\mathscr{A}})$ and $\operatorname{ann}^{(1)}_{D_n[s]}(f^s)$ can be taken to be homogeneous relative to the grading $x\rightsquigarrow 1, \partial\rightsquigarrow -1, s\rightsquigarrow 0$, we can assume P to be homogeneous. Then $\overline{P}$ is annihilated by $\tilde{E}=\sum_{i=1}^n \partial_i x_i$, while calculation reveals that also $\tilde{E}\cdot\overline{P}=\overline{P}\cdot(ds+n-\deg(P))$. Since a D_n-module supported at the origin is generated by its socle, $\tilde{b}(s)\cdot\operatorname{lcm}_{\overline{P}\in\Sigma}(ds+n-\deg(P))$ multiplies $\operatorname{ann}_{D_n}(f^s_{\mathscr{A}})$ into $\operatorname{ann}^{(1)}_{D_n[s]}(f^s_{\mathscr{A}})$. By construction and Kashiwara's theorem, the roots of this polynomial are in $\mathbb{Q}$. ☐

Remark 5.5. (1) The lemma and its proof is also correct for Saito-holonomic divisors with local quasi-homogeneities $\sum a_i x_i\partial_i$ that associate non-vanishing local degrees to the defining equation.

(2) Let $n=2$, $f=x^4+xy^4+y^5$. Then $\operatorname{ann}_{D_n[s]}(f^s)$ contains an operator of order two which is not multiplied into $\operatorname{ann}^{(1)}_{D_n[s]}(f^s)$ by any nonzero polynomial in s. Note that f is not Euler-homogeneous. ◇

Definition 5.6. For an arrangement $\mathscr{A}$, we let $b_{\log\mathscr{A}}(s)$ be the monic minimal polynomial in $\mathbb{Q}[s]$ that annihilates $\frac{\operatorname{ann}_{D_n[s]}(f^s_{\mathscr{A}})}{\operatorname{ann}^{(1)}_{D_n[s]}(f^s_{\mathscr{A}})}$.

Example 5.7 (The bracelet). Consider the central arrangement $\mathscr{A}$ in $X=\mathbb{C}^4$ defined by $f_{\mathscr{A}}=x_1x_2x_3(x_1+x_0)(x_2+x_0)(x_3+x_0)(x_1+x_2+x_0)(x_1+x_3+x_0)(x_2+x_3+x_0)$. The module $\operatorname{Der}_X(-\log_0 f_{\mathscr{A}})$ is generated by four derivations whose coefficients are all cubics in x. The module $\Omega^1_X(\log_0 f_{\mathscr{A}})$ has also four generators, and a minimal free resolution of the form $0\longrightarrow {R_4}^1\longrightarrow {R_4}^4\longrightarrow {R_4}^6\longrightarrow \Omega^1_X(\log_0 f_{\mathscr{A}})\longrightarrow 0$. In particular, $\mathscr{A}$ is not tame: $\operatorname{pdim}\Omega^1(\log f_{\mathscr{A}})=2$.

The only non-tame point is the origin. Hence, the ideal $\mathscr{L}_{f_\mathscr{A}}$ is prime in all points with at least one nonzero x-coordinate. However, it has a 4-dimensional component over the origin and in particular is not Cohen–Macaulay.

Outside the origin, $\operatorname{ann}_{D_n}(f_\mathscr{A}^s)$ is generated by derivations, but this is false at the origin: $\operatorname{ann}_{D_n}(f_\mathscr{A}^s)$ is generated by $\operatorname{Der}_X(-\log_0 f_\mathscr{A})$ and a (long) operator P of order 2 and degree 4 (in x). While we know no computer algebra system that can compute $\operatorname{ann}_{D_n}(f_\mathscr{A}^s)$ directly, one can check that the lead term (under the order filtration) of P, together with $\mathscr{L}_{f_\mathscr{A}}$, generates a prime ideal. It follows from Lemma 3.25 that $b_{\log \mathscr{A}}(s) = s + \frac{n+\deg_{(1,-1)}(P)}{d} = s + (4+4-2)/9 = s + 2/3$. $\diamond$

Remark 5.8. For an arrangement $\mathscr{A}$ and a complex number α, consider the smallest $i \in \mathbb{Z}$, such that $\operatorname{ann}_{D_n}(f_\mathscr{A}^{\alpha-i-j})$ is generated by derivations for all $j \in \mathbb{N}$. This "derivation index of f^α" should be related to the smallest root in $\alpha + \mathbb{Z}$ of the b-function of f on $D_n \cdot f^\alpha$.

From our theory, the following interesting question has a positive answer for all divisors that fit Theorem 3.26, but experimentally it is also correct for the bracelet (see Example 5.7) and the divisor $xy(x+y)(x+zy)$:

Question 5.9. Is $\operatorname{gr}_{(0,1)}(\operatorname{ann}_{D_n}(f^s))$ always prime?

The only candidate for the prime ideal is the defining ideal of the (closure in $T^*\mathbb{C}^n$ of the) union (over α) of all conormals to $\operatorname{Var}(f-\alpha)$.

5.3. **Bernstein–Sato polynomials and combinatorics.** The Bernstein–Sato polynomial $b_f(s)$ of a polynomial $f \in R_n$ is the monic generator of the ideal

$$\langle b_f(s)\rangle = D_n[s](f, \operatorname{ann}_{D_n[s]}(f^s)) \cap \mathbb{Q}[s].$$

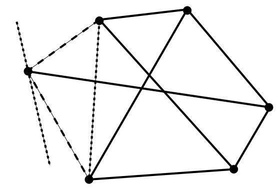
It is a difficult and long-standing problem to determine the Bernstein–Sato polynomial of an arrangement $f_\mathscr{A}$. While the case of a generic arrangement is completely understood (see [49, 43]), even in dimension three it is very mysterious how the singularities of the arrangement contribute to the roots of the Bernstein–Sato polynomial.

The Bernstein–Sato polynomial of an arrangement is more constrained than that of a general divisor. For example, all its roots have to be in the interval $(-2, 0)$, and one can read off the possible denominators of the roots directly from the arrangement by [45, Thms. 1+2]. A natural question in the context of arrangements is to what extent the intersection lattice governs the Bernstein–Sato polynomial. The following example indicates that one ought to ask a more refined question.

Example 5.10. Let $P_1, \ldots, P_6$ be six points in $\mathbb{P}^2$ and let $\mathscr{A}$ be the arrangement of 9 planes in $\mathbb{C}^3$ given as the union of the hyperplanes $G_{i,j}$ passing through P_i and P_j for $i - j = 1 \bmod 6$ or $i - j = 3 \bmod 6$. The arrangement is sketched as the union of the solid and dashed (but not the dotted) lines below. We assume that the points are sufficiently generic to make all 9 lines distinct, and so that the only triple points are the P_i. Let $J = \operatorname{Jac}(f_\mathscr{A})$.

Note that given three homogeneous linear forms $L_1, L_2, L_3 = aL_1 + bL_2 \in \mathbb{C}[x, y]$ then a curve defined by $\varepsilon(x, y)$ satisfies $\varepsilon \cdot L_3 \in \operatorname{Jac}(L_1 \cdot L_2 \cdot L_3)$ at $(0, 0)$ if and only if $\varepsilon(0, 0) = 0$ and the tangent of ε at $(0, 0)$ is the line defined by $aL_1 - bL_2$. We call this tangent line the *conjugate of* L_3 *relative to* L_1 *and* L_2.

For generic choices of the points, $H^0_{\mathfrak{m}}(R_3/J)$ is generated in degree 9 by six forms (one for each vertex) given as the union of the solid and dotted (but not the dashed) lines indicated on the right. The "tangential" dotted line on the far left is the conjugate of the big diagonal relative to the dashed lines at the vertex in question (and hence multiplies the big diagonal into the Jacobian there).



We investigate the question under what circumstances $H^0_{\mathfrak{m}}(R_3/J)$ contains the coset g_8 of the product of a cubic form c with $G_{1,6} \cdot G_{5,6} \cdot G_{4,5} \cdot G_{2,5} \cdot G_{3,6}$. Such curve must:

(1) pass through the points $P_7 = G_{1,2} \cap G_{3,4}$ and $P_8 = G_{1,4} \cap G_{2,3}$;
(2) pass through the points P_1, P_2, P_3, P_4 and in each of them have the correct tangent. (For example, the tangent at P_1 must be the conjugate line of $G_{6,1}$ relative to $G_{1,2}$ and $G_{1,4}$).

The vector space of cubics that passes through $P_1, P_2, P_3, P_4, P_7, P_8$ is typically 4-dimensional. If P_5 and P_6 are in generic position, the tangent conditions at P_1, P_2, P_3, P_4 on c are independent and so no suitable c exists.

Suppose $P_1, \ldots, P_5$ are in generic position and fixed. Then P_5 determines the necessary tangents of c at P_2 and P_4. View P_6 as movable, and choose the tangent direction for c at P_3 (hence a line on which P_6 should be located). This then determines a cubic with a tangent at P_1 that we cannot control. The conjugates of the tangents of c at P_1 (relative to $G_{1,2}$ and $G_{1,4}$) and P_3 (relative to $G_{2,3}$ and $G_{3,4}$) yield a unique point P_6 for which the conditions (1) and (2) above are satisfied.

Hence, with $P_1, \ldots, P_5$ fixed, the choices for P_6 that make the conditions (1) and (2) solvable form a curve q, parameterized by the choice of a line through P_3, and this curve meets any line through P_3 in exactly one point other than P_3. Choosing lines through P_3 that pass nearby P_1 one sees that P_1 lies on q and q is smooth there, with tangent given by the conjugate to the line through P_1 and P_3 relative to $G_{1,2}, G_{1,4}$. By symmetry, a similar statement holds at P_3.

Now pick a P_6 on q but generic otherwise, keep P_1, P_2, P_3, P_4 fixed, and move P_5 on q. The cubic c can then not change, since $P_1, \ldots, P_4, P_6$ pin it down. By the same reasoning as before, q passes through P_2, P_4 and is smooth in both points.

Finally, keep $P_1, \ldots, P_5$ fixed and degenerate P_6 into P_5. In order to have an element $G_{1,6} \cdot G_{5,6} \cdot G_{4,5} \cdot G_{2,5} \cdot G_{3,6}$ times a cubic in $H^0_{\mathfrak{m}}(R_3/J)$ it is necessary that $c \cdot G_{5,6}$ should

(1) pass through the points $P_7 = G_{1,2} \cap G_{3,4}$ and $P_8 = G_{1,4} \cap G_{2,3}$;
(2) pass through the points P_1, P_2, P_3, P_4 and in each of them have the correct tangent.

Because of the degeneration, the direction of $G_{5,6}$ can be chosen freely and this added degree of freedom makes the problem solvable.

In summa, for generic choices of $P_1, \ldots, P_5$ there is an element $g_8 = G_{1,6} \cdot G_{5,6} \cdot G_{4,5} \cdot G_{2,5} \cdot G_{3,6} \cdot c$, c a cubic, in $H^0_{\mathfrak{m}}(R_3/J)$ if and only if P_6 is a generic point on a curve q which satisfies: $P_1, \ldots, P_5 \in q$; q meets a generic line through P_3 in exactly one other point; q is smooth at P_3. In other words, q is the unique quadric through $P_1, \ldots, P_5$.

To be explicit, one can verify with *Macaulay 2* that one may take $G_{1,2} = 2x+y+z$, $G_{2,3} = x + y + z$, $G_{3,4} = 2x + 3y + 4z$, $G_{4,5} = z$, $G_{5,6} = x + 3z$, $G_{6,1} = y$, $G_{1,4} =$

$2x+3y+z$, $G_{2,5}=x$ and $G_{3,6}=x+2y+3z$. Then $q=2x^2+3xy+7xz+3yz+3z^2$ and $c=20x^3+68x^2y+73xy^2+24y^3+60x^2z+130xyz+65y^2z+51xz^2+54yz^2+13z^3$.

Pellikaan, and Van Straten and Warmt proved [48, Thm. 4.7] that since $\dim(J)=1$, $\deg(H^0_{\mathfrak{m}}(R_3/J))$ is an Artinian Gorenstein module. The proof identifies R_3 and R_3/J with their canonical modules and also, in [48, Prop. 3.6], the Koszul complex on the derivatives of $f_{\mathscr{A}}$ with its dual complex. Similar ideas are detailed in [23]. In the graded case, these identifications incur appropriate shifts and one obtains symmetry of the degrees of $H^0_{\mathfrak{m}}(R_3/J)$ about $3\deg(f)/2-3$. Hence a generator g_8 in degree 8 corresponds to a socle element in degree $2(3\cdot 9/2-3)-8=13$. (The Hilbert series of $H^0_{\mathfrak{m}}(R_3/J)$ in the example above is $T^8+4T^9+6T^{10}+6T^{11}+4T^{12}+T^{13}$; in the generic case, when the six points are not on a quadric, it is $4T^9+6T^{10}+6T^{11}+4T^{12}$.)

Since the singular loci of the arrangements are 1-dimensional, and since further the critical root $-16/9$ has no chance to be root of a local Bernstein–Sato polynomial outside the origin in any case, it follows from [43] that $-16/9$ is a root of $b_{f_{\mathscr{A}}}(s)$ if and only if 16/9 contributes to the pole order spectrum at the origin. However, by [23], one can decide this directly from the Hilbert series of $H^0_{\mathfrak{m}}(R_3/J)$, that of R_3/J and that of the Koszul homology groups of the derivatives f_1, f_2, f_3 on R_3. In particular, computations with *Macaulay 2* [29] can now easily confirm that in the degenerate case $-16/9$ must be a root of the Bernstein–Sato polynomial while it is not in the generic case. A detailed numerical discussion and an explanation of this conclusion has been given in [45, Remark 4.14] where the Bernstein–Sato polynomial has been determined entirely, but compare also [45, Remark 5.4].

An alternative computer-based method is the following. The arrangements are tame, no matter whether the vertices lie on a quadric. Thus, $\operatorname{ann}_{D_n[s]}(f^s_{\mathscr{A}})$ is generated by the Euler relation and the logarithmic vector fields that kill $f_{\mathscr{A}}$. By considering the $D_n[s]$-ideal generated by $E-9s$, $s+16/9$ and $\operatorname{Der}(-\log_0 f_{\mathscr{A}})$, V. Levandovskyy has confirmed with PLURAL [30] that the Bernstein–Sato polynomials differ in the generic and in the degenerate case.

Since the intersection lattice of $\mathscr{A}$ is the same for degenerate and generic $\mathscr{A}$, the Bernstein–Sato polynomial of an arrangement is not a function of the intersection lattice: differential invariants remember finer structure. ◇

It would seem natural to conclude that for an arrangement the Bernstein–Sato polynomial is not determined by linear information. However, it is our opinion that one should refine the intersection lattice and stick to linear data.

Definition 5.11. For a central arrangement $\mathscr{A}$, we define the *syzygetic intersection lattice* $L^+_{\mathscr{A}}$ as follows. Let $L^0_{\mathscr{A}}=L_{\mathscr{A}}$ and for $i\geq 1$ set $L^i_{\mathscr{A}}$ to be the set of all vector spaces that appear as nontrivial intersections of nontrivial sums of elements of $L^{i-1}_{\mathscr{A}}$. Now let S consist of the *syzygetic* elements V in $\bigcup_i L^i_{\mathscr{A}}$ that satisfy: $V\in L^i_{\mathscr{A}}$ is in S if there are elements $V_1,\dots,V_k$ in $L^{i-1}_{\mathscr{A}}$ such that $V=\sum_j V_j$ but $\dim(V)<\sum_j \dim(V_j)$. Then define $L^+_{\mathscr{A}}$ as the pair $(L_{\mathscr{A}},S)$.

The component S of $L^+_{\mathscr{A}}$ indicates syzygies (in the original sense of the word) between elements of $L_{\mathscr{A}}$.

In the degenerate case of Example 5.10, by the theorem of Pappus as refined by Pascal, the points that arise (say) as the intersections $G_{1,5}\cap G_{4,2}$, $G_{1,6}\cap G_{3,4}$, $G_{2,6}\cap G_{3,5}$ are *collinear*. (There are a maximum of 60 such Pascal lines in the degenerate case, but some of them may coalesce for certain arrangements). Thus, in the degenerate case, the affine cones over all Pascal lines would show up in $L^+(\mathscr{A})$. In contrast, the syzygetic lattice of all arrangements in the generic case contains no Pascal lines, so that $L^+_{\mathscr{A}}$ discriminates between the generic and the degenerate case although $L_{\mathscr{A}}$ does not.

5.4. **Strong Monodromy Conjecture.** Often, even very concrete questions regarding the root set ρ_f of the Bernstein–Sato polynomial of a polynomial are very hard to answer, even for arrangements. For example, the following seems essentially open:

Question 5.12. Suppose f is a homogeneous polynomial of degree d of embedding dimension n. Assuming that f is not the product of divisors in different sets of variables, under what circumstances is the number $-n/d$ a root of $b_f(s)$?

The n/d-Conjecture 1.3 of Budur, Mustaţă and Teitler [6] lists central indecomposable hyperplane arrangements as one important case where Question 5.12 is expected to have a positive answer. See [7] for positive results, including the case of reduced arrangements in dimension 3.

A sufficient condition for the n/d-Conjecture to hold is that the (coset of the) element $1 \in R_n$ be nonzero in the vector space $U_f \cong H^{n-1}(M_f, \mathbb{C})$, see [49, Thm. 4.12]. We do not know of significant classes of singularities where this topological version of the question has been answered. Our structural results about $\operatorname{ann}_{\mathscr{D}_X}(f^s)$ now enable us to prove that most arrangements satisfy Conjecture 1.3.

Theorem 5.13. *Let $\mathscr{A}$ be a central indecomposable arrangement (reduced or otherwise) with defining equation $f_{\mathscr{A}} = \prod_1^d L_i \in R_n$ on $X = \mathbb{C}^n$. Then $\operatorname{Der}_X(-\log_0 f_{\mathscr{A}})$ is contained in the ideal $D_n \cdot x$.*

If, in addition, $\operatorname{ann}_{D_n}(f^s_{\mathscr{A}})$ is generated by derivations then $b_{f_{\mathscr{A}}}(-n/d) = 0$.

Proof. As $f_{\mathscr{A}}$ is homogeneous, $\operatorname{Der}_X(-\log_0 f_{\mathscr{A}})$ and $\operatorname{ann}_{D_n}(f^s_{\mathscr{A}})$ are graded modules, where the degree of each x_i is 1 and that of each ∂_i is -1. The fact that $f_{\mathscr{A}}$ is indecomposable implies that there are no homogeneous logarithmic derivations of negative degree (that is, with constant coefficients). Namely, if $\delta = \sum_1^n c_i\partial_i$ were such a negative degree logarithmic derivation with $c_i \in \mathbb{C}$, then a judicious change of coordinates transforms δ into $\delta' = \partial_1$. But if this were to annihilate the transformed $\mathscr{A}$, then $\mathscr{A}$ should not be essential, and in particular decomposable.

We may restrict attention to homogeneous derivations. Observe that derivations of positive degree are always in $D_n \cdot x$ since in that case the degree of the coefficient exceeds the order of the operator. It follows that we only need to consider derivations δ that annihilate f and are of degree zero.

Such δ can be written as $\delta = x^T B\partial$ for a suitable square matrix $B \in \mathbb{C}^{n,n}$. Restricting its domain, one may interpret δ also as a linear operator on $\bigoplus_1^n \mathbb{C}\cdot x_i =: V$. Note that

$$\underbrace{\sum x_i b_{i,j}\partial_j}_{x^T B\partial} \bullet (\underbrace{\sum c_k x_k}_{x^T c}) \quad = \quad \sum x_i b_{i,j}\delta_{j,k}c_k = \sum x_i b_{i,j}c_j = x^T Bc$$

so that B represents both δ and this linear transformation on V.

As $\delta \bullet (f_{\mathscr{A}}) = 0$, δ is logarithmic along all L_k as well, so $\delta \bullet (L_k) = \gamma_k L_k$ with $\gamma_k \in \mathbb{C}$. In particular, the sum of the eigenspaces of δ on V is all of V, as $\mathscr{A}$ is essential. It follows that B is diagonalizable. Since a coordinate change $x' = Ax$

on $\mathbb{C}^n$ incurs the coordinate change $\partial_{x'} = A^{-T}\partial_x$ on $T^*(\mathbb{C}^n)_0$, it follows that diagonalizing B can be achieved by an appropriate coordinate change in $\mathbb{C}^n$.

In the right coordinates $\delta = \sum_1^n w_i x_i \partial_i$. Then, however, the condition $\delta \bullet (f_{\mathscr{A}}) = 0$ is a non-standard quasi-homogeneity of f in the sense that not all w_i can be equal. We show that this can't be happening.

Claim. An arrangement with a non-standard quasi-homogeneity δ is decomposable.

Proof. In a $\mathbb{Z}$-graded domain, the product of two nonzero expressions is δ-homogeneous precisely if the factors are. It follows that each defining hyperplane of the arrangement inherits the homogeneity δ. But a linear form is δ-homogeneous if and only if all variables occurring in that form have the same δ-degree. We have $w_i = \delta(x_i)/x_i \in \mathbb{C}$. Since δ is non-standard, the decomposition

$$\{x_1, \ldots, x_n\} = \{x_i \mid w_i = w_1\} \sqcup \{x_j \mid w_i \neq w_1\}$$

is nontrivial; the induced factorization shows that $\mathscr{A}$ is decomposable. $\square_{\text{Claim}}$

Returning to the proof of the theorem, indecomposable arrangements are not annihilated by logarithmic derivations of degree zero and it follows that $\operatorname{ann}_{D_n}(f_{\mathscr{A}}^s) \subseteq D_n \cdot x$.

Under the assumptions for the second claim we have proved thus the existence of a map

$$\frac{D_n}{D_n \cdot \operatorname{Der}_X(-\log_0 f_{\mathscr{A}})} \xrightarrow{=} \frac{D_n}{\operatorname{ann}_{D_n}(f_{\mathscr{A}}^s)} \twoheadrightarrow \frac{D_n[s]}{D_n[s] \cdot (x, E - ds)} \cong \frac{D_n}{D_n \cdot x}.$$

Pick a coset $\overline{P} = P + D[s] \cdot (x, E - ds)$ in the target of this map. Since $D_n[s] \cdot (x, E - ds)$ is $(1, -1)$-graded we may assume that P is $(1, -1)$-homogeneous. We calculate $s\overline{P} = \overline{Ps} = \overline{PE/d} = -\overline{nP/d}$ since $E + n \in D_n \cdot x$. It follows that s has minimal polynomial $s + n/d$ on the (nonzero!) quotient $D_n[s]/D_n[s] \cdot (x, E - ds)$ of $D_n[s]/D_n[s] \cdot (f_{\mathscr{A}}, E - ds, \operatorname{ann}_{D_n}(f_{\mathscr{A}}^s))$. So the Bernstein–Sato polynomial of $\mathscr{A}$ has to be a multiple of $(n + ds)$. $\square$

Recall the Strong Monodromy Conjecture from the introduction. For convenience, we introduce the following abbreviation from [47].

Notation 5.14. If $\operatorname{ann}_{\mathscr{D}_X}(f^s)$ is generated by derivations, we say that f *satisfies condition* (A_s).

Corollary 5.15. *An arrangement (reduced or otherwise) with property* (A_s) *satisfies the Strong Monodromy Conjecture. Specifically, this is true for tame arrangements.*

Proof. Arrangements allow for a combinatorial resolution of singularities, which makes the computation of the candidate poles of the topological zeta-function straightforward. This was used in [6] to show that the Strong Monodromy Conjecture for reduced arrangements boils down to the n/d-Conjecture for reduced arrangements. A closer inspection shows that this specific statement does a) not require reducedness of the divisor, and b) actually shows that for an arrangement $\mathscr{A}$ (reduced or otherwise) to satisfy the Strong Monodromy Conjecture it is sufficient to know that each indecomposable full subarrangement of $\mathscr{A}$ satisfies the n/d-Conjecture. Here we say that a subarrangement $\mathscr{A}'$ is *full* if for some flat of $\mathscr{A}$ the arrangement $\mathscr{A}'$ consists of exactly the hyperplanes passing through the flat, and the multiplicity of each hyperplane occurring in $\mathscr{A}'$ is its multiplicity in $\mathscr{A}$.

The property (A_s), as well as tameness, is inherited from any arrangement $\mathscr{A}$ with this property to each of its full subarrangements, as one sees from localizing at the respective flat. Suppose $\mathscr{A}$ has (A_s). By Theorem 5.13, all its indecomposable full subarrangements satisfy the n/d-Conjecture and so by the previous paragraph $\mathscr{A}$ satisfies the Strong Monodromy Conjecture.

The last claim of the corollary follows then from Theorem 5.2.

□

Remark 5.16. (1) By the previous result, all arrangements in dimension three satisfy the Strong Monodromy Conjecture. For reduced arrangements, this was shown in [7]. (In fact, [45] proves an even stronger form for reduced rank three arrangements: the product of $Z_f(s) \cdot b_f(s)$ is a polynomial). Their approach to showing that the n/d-Conjecture holds in the requisite cases is totally different from ours. We do not know how to generalize their method to the non-reduced situation.

(2) Using *Macaulay2* one verifies (by showing that the "long operator" in Example 5.7 is in $D_n \cdot x$) that the bracelet satisfies the n/d-Conjecture and hence the Strong Monodromy Conjecture, even though it fails (A_s).

(3) Consider $f_{\mathscr{A}} = (x^3+2y^3)(z^8-w^8)(x+z+w)$. Veys informs us that this arrangement does not have $-n/d = -1/3$ as pole in the topological zeta function, and $-1/3$ can also not be created as pole of more general integrals in the sense of [2]. Nonetheless, $f_{\mathscr{A}}$ is tame and so $\operatorname{ann}_{D_n}(f_{\mathscr{A}}^s)$ is generated by derivations wherefore the n/d-conjecture holds, and so $-1/3$ is a root of the Bernstein–Sato polynomial. ◇

6. Appendix: Logarithmic derivations under blow-ups

Notation 6.1. Let X be a smooth scheme over $\mathbb{C}$, with subschemes $C \subseteq Y \subseteq X$. Assume that C is smooth and denote by $\mathscr{I}_C \supseteq \mathscr{I}_Y$ the ideal sheaves. We say that a derivation on X is *logarithmic* along a subscheme V defined by $\mathscr{I}_V$ if $\delta(\mathscr{I}_V) \subseteq \mathscr{I}_V$.

Let

$$\pi\colon X' \longrightarrow X$$

be the (smooth) blow-up of X at C and denote $Y' \subseteq X'$ the total transform of Y.

Lemma 6.2. *In the context above, $\pi_*(\Omega^i_{X'}(\log \pi^*(Y))) = \Omega^i_X(\log Y)$.*

Proof. The statement is local in the base, so we may assume that X is smooth and affine, and represent $\Omega^i_X(\log Y)$ by their modules of global sections. We may assume further that $\operatorname{codim}_X(C) \geq 2$ as otherwise π is an isomorphism. Set $R = \Gamma(X, \mathscr{O}_X)$ and $I_C = \Gamma(X, \mathscr{I}_C)$.

Pulling back differentials along π is a linear faithful functor and there are $\mathscr{O}_X$-module inclusions $\underbrace{\Omega^i_X(\log Y)}_{=:M} \subseteq \underbrace{\Gamma(X', \Omega^i_{X'}(\log Y'))}_{=:M'} \subseteq \underbrace{K \otimes_R \Omega^i_X}_{=:M''}$, K denoting the field of fractions of R.

Let Q be defined by the exactness of $0 \longrightarrow M \longrightarrow M' \longrightarrow Q \longrightarrow 0$. Since M'', and hence also M', is torsion-free, the associated long exact sequence of local cohomology gives an injection $H^0_{I_C}(Q) \hookrightarrow H^1_{I_C}(M)$. Now M is a second syzygy, say the kernel of the map $F_1 \longrightarrow F_0$ between free modules. Then if L is the image of this map, long exact sequences again show that $H^1_{I_C}(M) = H^0_{I_C}(L)$ which vanishes as L sits in a free module and I_C has height at least two. We have shown that $H^0_{I_C}(Q) = 0$ but since Q is supported in C this forces $Q = 0$. □

We now consider lifting logarithmic derivations along a blow-up. While logarithmic derivations and logarithmic $(n-1)$-forms can be identified locally, globally $\Omega_X^{n-1}(\log Y)$ is really $\mathrm{Der}_X(-\log Y) \otimes \mathscr{O}_X(Y) \otimes \Omega_Y^n$ and so exhibits different behavior: the push-forward of the logarithmic derivations along $\pi^*(Y)$ is, in general, not all of the logarithmic derivations along Y.

Theorem 6.3. *In the setting of Notation 6.1,*

(1) an element of Der_X *lifts to* X' *if and only if it is logarithmic along* $\mathscr{I}_C$*; we have* $\pi_*(\mathrm{Der}_{X'}) = \mathrm{Der}_X(-\log \mathscr{I}_C)$*;*

(2) if Y *is* C*-saturated,* $\mathscr{I}_Y :_{\mathscr{O}_X} \mathscr{I}_C = \mathscr{I}_Y$*, then*

$$\pi_*(\mathrm{Der}_{X'}(-\log \pi^*(Y))) = \mathrm{Der}_X(-\log C) \cap \mathrm{Der}_X(-\log Y).$$

Proof. Again, we assume that X is smooth and affine, and that $\mathrm{codim}_X(C) \geq 2$. Set $R = \Gamma(X, \mathscr{O}_X)$ and write $C = \mathrm{Var}(I_C)$.

By shrinking X we can assume that $k = \mathrm{codim}_X(C)$ and $I_C = (f_1, \ldots, f_k)R \subseteq R$ where the f_i form a regular sequence in any order. Let $\delta \in \mathrm{Der}_X$ be logarithmic along C. Since $\delta \bullet (I_C) \subseteq I_C$, it induces a derivation $\tilde{\delta}$ of t-degree zero on the Rees ring

$$\mathscr{R} = \mathscr{R}(R, I_C) = \bigoplus_{i \in \mathbb{N}} (I_C t)^i.$$

Derivations of t-degree zero induce derivations on each homogeneous localization $\mathscr{R}[(tf_i)^{-1}]$, stabilize the degree-zero part, and then obviously agree on overlaps. Since $X' = \mathrm{Proj}(\mathscr{R})$, $\tilde{\delta}$ induces a global derivation $\delta' =: \pi^*(\delta)$ on $\mathscr{O}_{X'}$. The exceptional divisor E_π is defined by the ideal sheaf $I_C t\mathscr{R}$; as $\tilde{\delta} \bullet (I_C t\mathscr{R}) \subseteq I_C t\mathscr{R}$, δ' is logarithmic along E_π. The R-morphism $\delta \mapsto \pi^*(\delta)$ from Der_X to $\mathrm{Der}_{X'}(-\log E_\pi)$ is injective, since X and X' are smooth and agree outside C.

Now suppose additionally that $\delta \in \mathrm{Der}_X(-\log Y)$; then $\tilde{\delta}$ preserves the extension of I_Y to $\mathscr{R}$, and hence δ' is logarithmic along the total transform of Y. Since I_C is prime and $I_Y :_R I_C = I_Y$, a dense open set of each component of the proper transform Y' is outside E_π. It follows that δ' is also logarithmic along Y' and so $\mathrm{Der}_X(-\log Y) \cap \mathrm{Der}_X(-\log C)$ is a submodule of $\Gamma(X', \mathrm{Der}_{X'}(-Y'))$ and of $\Gamma(X', \mathrm{Der}_{X'}(-\pi^*(Y)))$.

Conversely, let δ' be a global derivation on X' and let U_i be the standard open set $\mathrm{Spec}(\mathscr{R}_i)$ in X' where $\mathscr{R}_i = (R[tf_1, \ldots, tf_k, (tf_i^{-1})])_0 = R[f_1/f_i, \ldots, f_k/f_i]$. By definition, δ' induces a derivation on each $\mathscr{R}_i$ which we also denote δ'.

Since R is a domain, $R \subseteq \mathscr{R}_i \subseteq R[1/f_i]$ and so $\delta' \bullet (R) \subseteq \bigcap_i \mathscr{R}_i \subseteq \bigcap_i R[1/f_i]$. The latter is the ideal transform of R with respect to I_C, and since the depth of I_C on R is at least 2 we have $(\bigcap R[1/f_i])/R = H^1_I(R) = 0$. Hence δ' is a derivation on R which we denote δ. We must show that it is logarithmic along the center of the blow-up. Note that $\pi^*(\delta) = \delta'$ again.

Lemma 6.4. *Let* R *be a domain,* $f_1, \ldots, f_k, g$ *a regular sequence in* R*, and* δ' *a derivation on both* R *and* $R[f_1/g, \ldots, f_k/g]$*. Let* I *be the* R*-ideal generated by* $f_1, \ldots, f_k$*. Then* $\delta' \bullet (I) \subseteq I + Rg$*.*

Proof. By hypothesis, $\delta' \bullet (f_i/g) = \delta' \bullet (f_i)/g - f_i \delta' \bullet (g)/g^2$ can be written as $\sum_{j=0}^r P_j(f_1/g, \ldots, f_k/g)$ for suitable homogeneous polynomials $P_0, \ldots, P_r \in R[x_1, \ldots, x_k]$, P_j being of degree j. Choose such presentation with r minimal.

We show first that one can assume $r \leq 2$. Indeed, if $r > 2$ clear denominators to see that $P_r(f_1, \ldots, f_k) \in Rg$. By Lemma 6.5 below, $P_r(f_1, \ldots, f_k) = Q_r(f_1, \ldots, f_k)$

for a suitable homogeneous polynomial $Q_r(x) = g\sum_M q_M x^M \in gR[x_1,\ldots,x_k]$ using multi-index notation $x^M = \prod_i x_i^{m_i}$ with $|M| = \deg(P_r) = r$. Then, abbreviating $f_1,\ldots,f_k$ to f, and $f_1/g,\ldots,f_k/g$ to f/g,

$$P_r(f/g) = \sum_M q_M \cdot g \cdot (f/g)^M = \sum_M q_M \cdot g^{1-|M|} \cdot f^M.$$

For each multi-index M, let $i(M)$ be some index with $M_i > 0$ and denote e_i the unit vector in direction i. Then $\sum_M q_M g^{1-|M|} f^M = \sum_M q_M f_{i(M)} g^{1-|M|} f^{M-e_{i(M)}}$ can be viewed as evaluation of $\sum_M q_M f_{i(M)} x^{M-e_{i(M)}}$ at f/g. The latter is a polynomial of degree $r-1$. And so the presentation $\sum_{j=0}^r P_i(f_1/g,\ldots,f_k/g)$ for $\delta'(f_i/g)$ didn't use minimal r.

Hence, we may assume that $r \le 2$. In that case, we obtain, clearing denominators,

$$\delta' \bullet (f_i)g - \delta' \bullet (g)f_i = P_0(f)g^2 + P_1(f)g + P_2(f).$$

In particular, $\delta' \bullet (f_i)g - P_0(f)g^2 = P_1(f)g + P_2(f) + \delta' \bullet (g)f_i \in I$. Since g is regular on R/I, $\delta' \bullet (f_i) - P_0(f)g \in I$ and so $\delta' \bullet (f_i) \in R(f,g)$. This finishes the proof of Lemma 6.4 □

Now return to the proof of the Theorem. Lemma 6.4 implies, looking at U_i, that $\delta \bullet (f_i) \in I$ for each i. Hence, δ is logarithmic along I_C, and therefore δ' is logarithmic along E_π.

If δ' is logarithmic along Y' then δ' preserves each $I_Y \cdot \mathscr{R}_i$, and hence δ sends I_Y into $\bigcap(I_Y \cdot \mathscr{R}_i)$. However, $\bigcap(I_Y \cdot \mathscr{R}_i)/I_Y = H^1_{I_C}(I_Y)$. Since C is cut out by a regular sequence of length at least 2 in R, $H^1_{I_C}(I_Y) = H^0_{I_C}(R/I_Y)$ and this last module vanishes since by hypothesis R is $\mathscr{I}_C$-torsion-free. It follows that δ is logarithmic along Y and the theorem follows. □

Lemma 6.5. *Let R be a domain, and let $f_1,\ldots,f_m,g_1,\ldots,g_{m'}$ be a regular sequence in every order on R. Write $f = f_1,\ldots,f_m$ and $g = g_1,\ldots,g_{m'}$.*

Suppose $P(x) \in R[x_1,\ldots,x_m]$ is homogeneous of degree k and satisfies $P(f) \in Rg$. Then there exists $Q \in R[x_1,\ldots,x_m]$, homogeneous of degree k, with $P(f) = Q(f)$ and $Q(x) \in gR[x]$.

Proof. (1) If $m = 1$ then we have $rf^k \in Rg$ and so regularity implies that the lemma holds in that case.

(2) If P is linear, $\sum r_i f_i \in Rg$ and so $r_m \in R(g, f_{<m})$ can be written as $\sum b_j g_j + \sum_{i<m} a_i f_i$. Rewriting, we obtain $P(f) = \sum_{<m} r_i f_i + r_m f_m = \sum_{<m} r_i f_i + \sum_{<m} f_m a_i f_i + \sum f_m b_j g_j = \sum_{i<m}(f_m a_i + r_i)f_i + \sum f_m b_j g_j \in R(f_{<m,g})$. By induction on m, there is a linear polynomial $Q'(x_1\ldots,x_{m-1})$ over R with coefficients in Rg which when evaluated at $f_{<m}$ yields $\sum_{i<m}(f_m a_i + r_i)f_i$. But then $P(f) = Q'(f_{<m}) + f_m \sum b_j g_j$ and the lemma follows in this case.

(3) The general case. We use induction on m. Consider the stratification of the monomials in m variables of degree k into the following subsets: $S_m = \{x_m^k\}$, $S_j = \{\text{monomials in } x_{\ge j}\} \smallsetminus S_{j+1}$ for $0 < j < n$. So S_j is the set of monomials in $R[x_j,\ldots,x_m]$ divisible x_j. We are going to construct a sequence of polynomial identities

$$(j): \qquad P^{(j)}(x) = \underbrace{\sum_{\ell<j} P^{(j)}_\ell(x)}_{P^{(j)}_{<j}(x)} + P^{(j)}_j(x) + \underbrace{\sum_{\ell>j} P^{(j)}_\ell(x)}_{P^{(j)}_{>j}(x)}$$

such that the following hold: $P^{(j)}_\ell(x)$ is a sum of monomials indexed by S_ℓ; $P^{(j)}_{>j}(x) \in gR[x_{j+1},\ldots,x_m]$; $P^{(j)}(f) = P(f)$ for all j.

For $j = m$, take for $P^{(m)}_m(x)$ the part of $P(x)$ indexed by S_m, and put $P^{(m)}_{>m}(x) = 0$ and $P^{(m)}_{<m}(x) = P(x) - P^{(m)}_m(x)$.

For any j, $P^{(j)}_{<j}(x)$ is automatically in the ideal generated by $x_{<j}$, so evaluation at $x = f$ gives $P^{(j)}_j(f) = P(f) - P^{(j)}_{<j}(f) - P^{(j)}_{>j}(f) \in R(g, f_{<j})$.

Given identity (j) we now construct identity $(j-1)$ with the corresponding properties. As x_j divides $P^{(j)}_j(x)$, $P^{(j)}_j(f)/f_j \in R$. Then $R(g, f_{<j}) \ni P^{(j)}_j(f) = f_j \cdot (P^{(j)}_j(f)/f_j)$ implies by regularity that $P^{(j)}_j(f)/f_j \in R(g, f_{<j})$. As the polynomial $P^{(j)}_j(x)/x_j \in R[x_{\geq j}]$ is homogeneous of degree $k-1$, the inductive hypothesis asserts the existence of a homogeneous polynomial of degree $k-1$ in $x_j,\ldots,x_m$ with coefficients in $R(g, f_{<j})$ and which evaluates at f to $P^{(j)}_j(f)/f_j$. Explicitly, $P^{(j)}_j(f)$ is the value at $x = f$ of the sum of one polynomial $P^{(j)}_{j,g}(x) \in gR[x]$ and another polynomial in $f_{<j}R[x]$, both homogeneous of degree $k-1$. The latter polynomial can be changed (without affecting its value at $x \rightsquigarrow f$) into a polynomial $P^{(j)}_{j,<}(x) \in x_{<j}R[x]$ which is then supported in $S_{<j}$. Moving terms, if necessary, from $P^{(j)}_{j,g}(x)$ to $P^{(j)}_{j,<}(x)$ one may assume that $P^{(j)}_{j,g}(x) \in R[x_{>j-1}]$. Now update identity (j) as follows: set $P^{(j-1)}_{<j-1}(x)$ to be the terms in $P^{(j)}_{<j}(x)+P^{(j)}_{j,<}(x)$ supported in $S_{<j-1}$; let $P^{(j-1)}_{j-1}(x)$ be the other terms in this sum; set $P^{(j-1)}_{>j-1}(x) = P^{(j)}_{>j}(x) + P^{(j)}_{j,g}(x)$. The stipulated conditions then hold for the new display.

It follows that from identity (m) above we can proceed to identity (0). However, then $P^{(0)}(x) = P^{(0)}_{>0}(x) \in gR[x]$. □

We now record the existence of an embedded resolution of singularities that is particularly well adapted to computing with logarithmic vector fields.

Proposition 6.6. *For every divisor Y on X, there is an algorithm for construction of an embedded resolution of singularities that is a composition of blow-ups such that at each step the center of the blow-up is smooth and a union of logarithmic strata of the total transform of Y under the previous blow-ups.*

Proof. By [51], there is an embedded resolution of singularities $\pi\colon X' \longrightarrow X$ that is a sequence of blow-ups $\pi = \pi_k \circ \ldots \circ \pi_1$ with smooth centers that furthermore is functorial: for any analytic isomorphism $\iota\colon X \longrightarrow X$ there is an analytic isomorphism $\tilde{\iota}\colon X' \longrightarrow X'$ such that $\pi \circ \tilde{\iota} = \iota \circ \pi$.

Take $\delta \in \operatorname{Der}_X(-\log Y)$ and choose $\mathfrak{x}_0 \in X$ with $\delta(\mathfrak{x}_0) \neq 0$. By the Picard–Lindelöf theorem, there is a foliation of integrating curves $\gamma_{\mathfrak{x}}(t)$ for δ at all $\mathfrak{x}$ near $\mathfrak{x}_0$: $\gamma_{\mathfrak{x}}(0) = \mathfrak{x}$ and $\frac{\mathrm{d}}{\mathrm{d}t}(\gamma_{\mathfrak{x}}(t))$ is δ evaluated at $\gamma_{\mathfrak{x}}(t)$ for small t. It follows that the assignment $\Phi(\mathfrak{x}, t)\colon \mathfrak{x} \mapsto \gamma_{\mathfrak{x}}(t)$ is well-defined for $\mathfrak{x}$ sufficiently near $\mathfrak{x}_0$, and for all $t \ll 1$. Since δ is analytic and $\gamma_{\mathfrak{x}}(t)$ is defined via an integral, Φ is analytic.

As δ is logarithmic along Y, $\mathfrak{x} \in Y$ implies $\gamma_{\mathfrak{x}}(t) \in Y$. Hence, $\Phi(-, t)$ is a local automorphism of the pair (X, Y) near $\mathfrak{x} \in Y$ for $t \ll 1$. If $\pi_1\colon X'_1 \longrightarrow X$ is the first blow-up in the resolution, then by functoriality $\Phi(-, t)$ lifts to an analytic family of local automorphisms of (X'_1, π^*Y). This is only possible if the center C_1 of the blow-up is stable under δ. This being true for all logarithmic vector fields along Y, C_1 must be a union of logarithmic strata of Y. The proposition follows by iterating the argument. □

Corollary 6.7. *If $Y \subseteq X$ is a divisor in a smooth $\mathbb{C}$-scheme then there is a resolution of singularities $\pi\colon X' \longrightarrow X$ such that $\pi_*(\mathrm{Der}_{X'}(-\log \pi^*(Y))) = \mathrm{Der}_X(-\log Y)$ and $\pi_*(\Omega^i_{X'}(\log \pi^*(Y))) = \Omega^i_X(\log Y)$.*

Proof. For any divisor Y_i and any smooth center $C_i \subseteq Y_i$ of codimension 2 or more on smooth X_i the ideal quotient $\mathscr{I}_{Y_i} :_{\mathscr{O}_{X_i}} \mathscr{I}_{C_i}$ equals $\mathscr{I}_{Y_i}$. Thus Theorem 6.3 applies in each step of the resolution obtained in Proposition 6.6. The final claim follows from Lemma 6.2. □

Not every logarithmic stratum will appear as a center in a resolution:

Example 6.8. Let $Y = \mathrm{Var}(xy(x+y)(x+ty)) \subseteq \mathbb{C}^3$. Then every point of the t-axis is a logarithmic stratum. In any reasonable resolution of the pair $(\mathbb{C}^3, Y)$, of all points on the t-axis, the only zero-dimensional canonical Whitney strata that feature as blow-up centers are $(0,0,0)$ and $(0,0,1)$. ⋄

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