BIVARIATE RATIONAL (A-)HYPERGEOMETRIC FUNCTIONS & RESIDUES

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GOAL:

UNDERSTAND, CLASSIFY, IDENTIFY ALL (STABLE) RATIONAL A-HYPERG. FUNCTIONS f, i.e. ALL (STABLE) MONODROMY INVARIANT SOLUTIONS OF $H_A(\beta)$ (for regular A)

KNOWN:

$$f = \frac{p}{q} \qquad q = \prod_{\substack{A' \subseteq A \\ \text{facial}}} D_{A'}^{m_{A'}}, \quad m_{A'} \in \mathbb{N}$$

since $\left(\prod_{A'} D_{A'} = 0\right) = \operatorname{sing}(H_A(\beta))$ [G.K.Z.]

OUR SETTING:

- $A \in \mathbb{Z}^{d \times (d+2)}$, $n = d+2$, $\mathbb{Z} \cdot A = \mathbb{Z}^d$

  $(1, \ldots, 1) \in$ row span (A)

  codimension two configuration

- A is affinely equivalent to a Cayley configuration

$$\bigcup_{i=0}^{n} \{e_i\} \times A_i \subseteq \mathbb{Z}^{r+1} \times \mathbb{Z}^r$$

$A_0, \ldots, A_r \subseteq \mathbb{Z}^r$ finite
$e_0, \ldots, e_r$ canonical basis of $\mathbb{Z}^{r+1}$

which is essential, i.e. if $\forall I \subseteq \{0, \ldots, r\}$

$$\dim \left(\sum_{i \in I} A_i \right) \geq |I|$$

KNOWN: A admits a stable hypergeometric function solution iff A is affl. eq to an essential Cayley configuration
[CDS, 01] [CDD, 99] [CDS, 02] [CD, 04] [CDRV, 08]

OBS: $f_A(y, t) = \sum_{i=0}^{r} y_i f_i(t)$, $f_i = f_{A_i} = \frac{\partial f_A}{\partial y_i}$ $i = 0, \ldots, r$

$$D_A(f) = \mathrm{Res}_{A_0, \ldots, A_r}(f_0, \ldots, f_r)$$ DEPENDS ON ALL THE VARIABLES

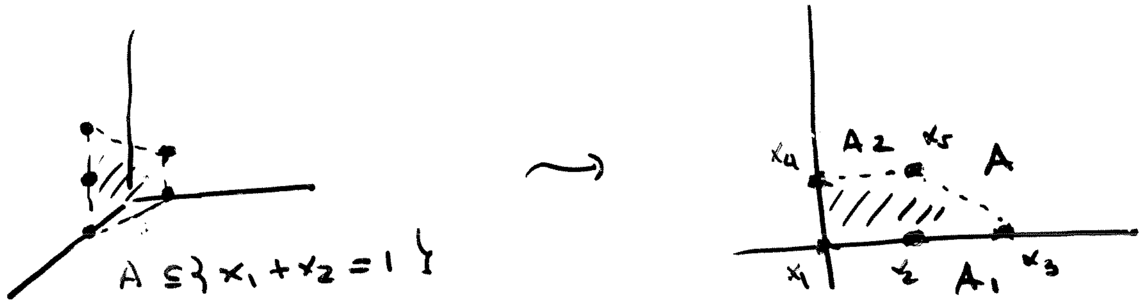
Example: $r = 1$

$A_1 = \{0, 1, 2\}$ $\qquad$ $A_2 = \{0, 1\}$

$A = \{(1,0,0), (1,0,1), (1,0,2), (0,1,0), (0,1,2)\}$

$e_1 = (1,0)$ $\qquad$ $e_2 = (0,1)$

$A = e_1 \times A_1 \cup e_2 \times A_2$



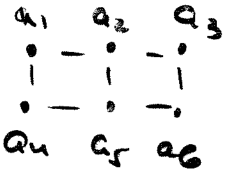
Cayley configurations are very special configurations

If A is an essential Cayley configuration of codimension 2, then it is aff eq. to the Cayley config. associated to r binomials (in r linearly independent directions in $\mathbb{Z}^r$) and 1 trinomial.

How to construct rational hypergeom. functions?

Example $n=6,\ d=3$

$$A = \begin{pmatrix} 1&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 0&1&2&0&1&2 \end{pmatrix}$$



$$f_0 = x_1 + x_2 t + x_3 t^2 \qquad F_0 = x_1 s^2 + x_2 st + x_3 t^2$$

$$f_1 = x_4 + x_5 t + x_6 t^2 \qquad F_1 = x_4 s^2 + x_5 st + x_6 t^2$$

$$D_A(x) = \text{Resultant}_{2,2}(f_0, f_1) =$$

$$x_1^2 x_6^2 - x_1 x_2 x_5 x_6 - 2 x_1 x_3 x_4 x_6 + \cdots + x_3^2 x_4^2$$

Let $(f_1 = 0) = \{\xi_1, \xi_2\}$ $\quad \xi_i$ algebraic functions of x_0, x_1, x_2

$$\varphi_i(x) = \frac{\xi_i}{f_0(\xi_i) f_1'(\xi_i)} = \operatorname{Res}_{\xi_i}\left(\frac{t/f_0}{f_1}\, dt \right) \quad i = 1,2$$

$\varphi(x) = \varphi_1(x) + \varphi_2(x)$ is rational

Each φ_i (and so φ) is A-hypergeometric

Moreover

$$\varphi(x) = \frac{1}{4} \frac{\text{normalf}_{F_0 F_1}(t\cdot s)}{\text{normalf}_{F_0,F_1}(J_F)} = \frac{x_1 x_6 - x_3 x_4}{D_A}$$

Toric residue

Cox, C.E.D., C.D.

Computed by a GB Computation

$$\| \qquad \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma \frac{t\, s\, dt\, ds}{F_0 F_1}$$

Γ compact 2 cycle in $\mathbb{C}^2$

CONSTRUCTION OF STABLE RAT. HYP. FUNCTIONS

$$A = e_0 \times A_0 \cup \dots \cup e_r \times A_r \qquad \text{essential}$$

$$f_i = \sum_{a \in A_i} x_a t^a \quad i = 0, \dots, r \qquad t = (t_1, \dots, t_r)$$

$$a \in \operatorname{int}(\Delta_0 + \dots + \Delta_r) \cap \mathbb{Z}^r$$

$$\Delta_i = \underset{A_i}{\text{convex hull}} \qquad \mathbb{Z}A_0 + \dots + \mathbb{Z}A_r = \mathbb{Z}^r$$

$\forall\ j = 0, \dots, r$

$$\operatorname{Res}_f(t^a) = (-1)^j \sum_{\xi \in \bigcap_{k \neq j} (f_k = 0) \subseteq (\mathbb{C}^*)^r} \operatorname{Res}_\xi \left(\frac{t^a / f_j}{f_0 \cdots \hat{f_j} \cdots f_r} \frac{dt}{t} \right)$$

$$= \sum_\xi \xi^a \Big/ f_j(\xi)\, J^T_{f_0 \cdots \hat{f_j} \cdots f_r}(\xi)$$

Has an <u>integral</u> representation

is independent of j and defines a stable rational A-hyperg. function

[CCD, 97] [CDS, 01] [AS, 96]

$$\operatorname{Res}_\xi = \operatorname{Res}_{\xi, f_0, \dots \hat{f_j}, \dots f_r} \qquad \underline{\text{Grothendieck}}\ \underline{\text{joint}}\ \underline{\text{residue}}$$

- It can be computed via GB computations
  - Denominator is D_A [CDS, 97]
  - Essential $=$ $MV(A_0, \dots, \hat{A_j}, \dots A_r)$ is > 0 $\forall j$

    i.e. $\bigcap_{k \neq j} (f_k = 0) \neq \emptyset$
- $\xi = \xi(x)$ local branch of a common root

MORE GENERALLY

$A = e_0 \times A_0 \cup \dots \cup e_r \times A_r$ essential

$f = \sum_{a \in A_i} x_a t^a \qquad i = 0, \dots, r$

$b_0, \dots, b_r \in \mathbb{N}, \quad a \in \operatorname{int}\left(\sum_{i=0}^{r} b_i \Delta_i \right)$

Δ_i = convex hull of A_i

$\mathbb{Z} A_0 + \dots + \mathbb{Z} A_r = \mathbb{Z}^r$

TORIC RESIDUE:

$j \in \{0, \dots, r\}$

$$\sum_{\xi \in \left(\bigcap_{k \neq j} f_k = 0 \right) \subseteq (\mathbb{C}^*)^r} \operatorname{Res}_\xi \left(\frac{t^a / f_j^{b_j}}{f_0^{b_0} \cdots \hat{f_j} \cdots f_r^{b_r}} \frac{dt}{t} \right)$$

is independent of j, rational, stable, A-hyperg function with homog $(-b_0, \dots, -b_r, -a)$

Denominator: Product of powers of the resultant $\operatorname{Res}_{A_0, \dots, A_r}(f_0, \dots, f_r) = D_A(f)$ (always present) and possibly facet resultants ass. to subsets of $f_0, \dots, f_r$

MAIN THEOREM:

Let A be a codimension two <u>essential</u> Cayley configuration

$$\beta \in (\underline{-\, \mathrm{Pos}(A)})^{\circ} \cap \mathbb{Z}A$$
$$\mathbb{Z}A = \mathbb{Z}^d$$

Euler-Jacobi cone

$$\left(\text{equiv } \beta \in A \cdot (\mathbb{Q}_{<0})^n \cap \mathbb{Z}^d\right)$$

Then
$$\dim_{\mathbb{C}} \begin{pmatrix} \text{rational (stable)} \\ A\text{-hyp. functions} \\ \text{of } A\text{-homog } \beta \end{pmatrix} = 1$$

and the space is spanned by the <u>toric residue</u>

COROLLARY:

A as above, $\beta \in \mathbb{Z}^d = \mathbb{Z}A$

f a <u>rational</u> A-hyp. function of homogeneity β

There exists a derivative α such that
$$\partial^{\alpha} f = 0 \qquad f \text{ unstable}$$
$$\text{or } \partial^{\alpha} f = \text{non zero multiple of a toric residue} \qquad f \text{ stable}$$

MAIN IDEAS IN THE PROOF:

- Laurent series expansions of A-hyp fn's

$$\Updownarrow$$

Minimal regions in the complement of an hyperplane arrangement

$\begin{cases} \text{in } B\text{-space (Gale dual) dim} = 2 \\ \text{in } A\text{-space 2 plane in dim} = n \end{cases}$

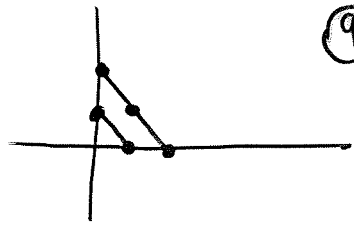
- Rational functions $f = \frac{p}{q}$ have Laurent series expansions coming from the "geometric series trick" at any vertex $\underline{v}$ of the Newton polytope of q

- Which Laurent series expansions are rational?

Obs: Laurent series expansions (convergent) correspond to functions locally invariant by the monodromy which linear combinations are globally invariant?

EXAMPLE

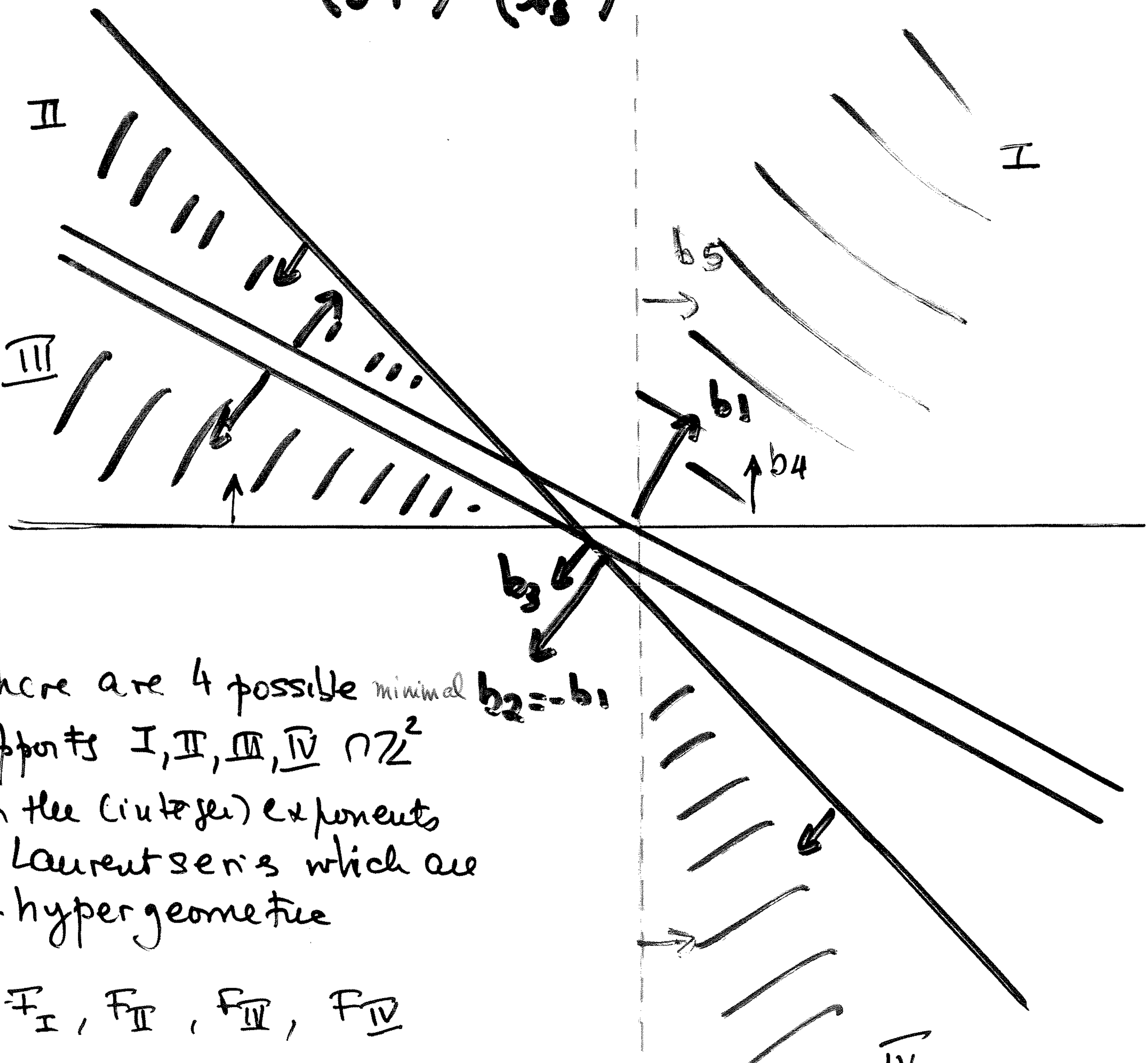
$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 2 \end{pmatrix}$



(Choice of) Gale dual

$B = \begin{pmatrix} 1 & 2 \\ -1 & -2 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_5 \end{pmatrix}$

$\beta = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} = A \cdot \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$



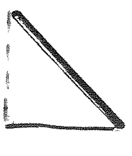
There are 4 possible minimal $b_2 = -b_1$
supports $I, II, \underline{III}, \underline{IV} \cap \mathbb{Z}^2$
for the (integer) exponents
of Laurent series which are
A-hypergeometric

$\leadsto F_I, F_{II}, F_{\underline{III}}, F_{\underline{IV}}$

ONLY F_I & $F_{\underline{II}}$, or $F_{\underline{II}}$ & $F_{\underline{III}}$, or $F_{\underline{III}}$ & F_{IV}
have a common domain of convergence

TORIC RESIDUE $\quad f(x) = \dfrac{x_1}{x_3 x_2^2 - x_1 x_2 x_4 + x_1^2 x_3}$

NP(denominator):



$= \begin{cases} \text{in one open set} & F_I \\ \text{in another one} & F_{IV} \\ \text{''} \quad \text{''} \quad \text{''} & -F_{II} - F_{\underline{III}} \end{cases}$

$F_{\underline{II}}$ & $F_{\underline{III}}$ are NOT rational