Degree bounds for Gröbner bases in algebras of solvable type

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- Monoid: for $x = (x_1, \dots, x_N)$ let $x^\diamond = \{x^\alpha \mid \alpha \in \mathbb{N}^N\}$ with multiplication $x^\alpha * x^\beta = x^{\alpha+\beta}$ and an admissible ordering $\leq$.
- Algebra: $R = K\langle x \rangle = K\langle X \rangle / I(\mathcal{R})$ over a field K generated by x with a commutation system $\mathcal{R}$.

$$R_{ij} = X_j X_i - c_{ij} X_i X_j - P_{ij} \in K\langle X \rangle$$

$$\text{where } 0 \neq c_{ij} \in K \text{ and } P_{ij} \in \bigoplus_\alpha K X^\alpha \text{ for } 1 \leq i < j \leq N.$$

(Here: $K\langle X \rangle$ is free algebra on letters X_i, $1 \leq i \leq N$.)

- Affine algebra: $x^\diamond$ is a K-basis for R.
- Algebra of solvable type: $\operatorname{lm} P_{ij} < X_i X_j$.

Example (Weyl algebra)

$R = A_n(K) = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$ with relations

$$\begin{array}{lll} x_j x_i = x_i x_j, & \partial_j \partial_i = \partial_i \partial j & \text{for } 1 \leq i < j \leq n, \\ \partial_j x_i = x_i \partial_j & & \text{for } 1 \leq i, j \leq n,\ i \neq j, \\ \partial_i x_i = x_i \partial_i + 1 & & \text{for } 1 \leq i \leq n. \end{array}$$

If R is of solvable type, then

- for non-zero $f, g \in R$, $\operatorname{lm}(f \cdot g) = \operatorname{lm}(f) * \operatorname{lm}(g)$;
- Gröbner bases techniques are possible.

Homogenization:

- R is called homogeneous if the commutation system $\mathcal{R}$ is homogeneous.
- R is called quadric if the $\deg P_{ij} \leq 2$.
- Quadric → homogeneous: append a homogenizing variable.

Example (Homogenization of the Weyl algebra)

$A_n^*(K) = \langle x_1, \dots, x_n, \partial_1, \dots, \partial_n, t \rangle$ with relations

$$\begin{array}{lll} x_j x_i = x_i x_j, & \partial_j \partial_i = \partial_i \partial_j & \text{for } 1 \leq i < j \leq n, \\ \partial_j x_i = x_i \partial_j & & \text{for } 1 \leq i, j \leq n,\ i \neq j, \\ \partial_i x_i = x_i \partial_i + t^2 & & \text{for } 1 \leq i \leq n, \\ x_i t = t x_i, & \partial_i t = t \partial_i & \text{for } 1 \leq i \leq n. \end{array}$$

Degree bounds on the elements of reduced Gröbner bases (commutative case):

- Lower degree bound: doubly exponential in N (Mayr-Meyer; 1982).
- Upper degree bound: doubly exponential in N
  - “General coordinates” arguments (Bayer, Möller-Mora, Guisti; 1980’s).
  - Cone decompositions (Dubé-Yap, 1980’s; Dubé, 1991).

What was believed for the non-commutative case, e.g., Weyl algebra:

- The upper bound should be larger.
- It is easy to prove that the bound is the same as for polynomials.

Recently obtained doubly exponential bounds:

- Chistov and Grigoriev, *Complexity of Janet basis of a D-module,* (2007).
- Aschenbrenner and L. (2007)

For K-subspaces spanned by monomials:

- **Monomial cone**: for a pair (w, y) with $w \in x^\diamond$ and $y \subseteq x$,

$$C(w, y) = \mathrm{Span}_K(w * y^\diamond).$$

- A collection of cones $\mathcal{D}$ is a **monomial cone decomposition** of a K-space M if $C(w, y) \subseteq M$ for every $(w, y) \in \mathcal{D}$ and

$$M = \bigoplus_{(w,y)\in\mathcal{D}} C(w, y).$$

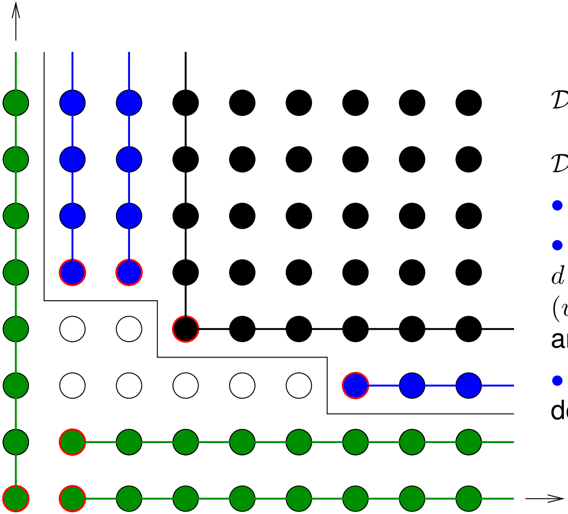
For R, a homogeneous algebra of solvable type:

- **Cone**: a triple (w, y, h), where $h \in R$ is homogeneous.

$$C(w, y, h) := C(w, y)h = \{gh : g \in C(w, y)\} \subseteq R.$$

- $\mathcal{D}$ is a **cone decomposition** of a homogeneous K-subspace M of R if $C(w, y, h) \subseteq M$ for every $(w, y, h) \in \mathcal{D}$ and $M = \bigoplus_{(w,y,h)\in\mathcal{D}} C(w, y, h)$.

For an ideal $I \subset R$ with a Gröbner basis G split $R = \mathrm{nf}_G(R) \oplus I$ and construct a cone decomposition for both parts.

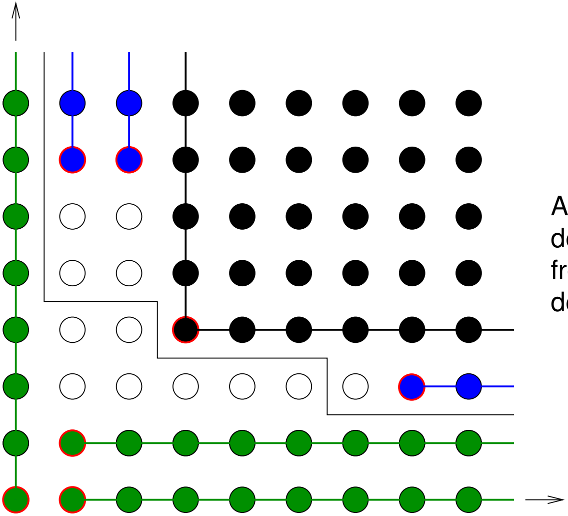


$$\mathcal{D}^+ := \{(w, y) \in \mathcal{D} : y \neq \emptyset\}$$

$\mathcal{D}$ is d-standard if

- $\deg(w) \geq d$ for all $(w, y) \in \mathcal{D}^+$;
- $\forall (w, y) \in \mathcal{D}^+$ and d' with $d \leq d' \leq \deg(w)$ there is some $(w', y') \in \mathcal{D}^+$ with $\deg(w') = d'$ and $\#y' \geq \#y$.
- There exists a 0-standard decomposition for $\mathrm{nf}_G(R)$.

A cone decomposition $\mathcal{D}$ is exact if $\mathcal{D}$ is d-standard for some d and $\forall d'$ there is at most one $(w, y, h) \in \mathcal{D}^+$ with $\deg(w) + \deg(h) = d'$.



An exact d-standard decomposition can be obtained from a d-standard cone decomposition.

Let $d_{\mathcal{D}}$ be the smallest d, for which $\mathcal{D}$ is d-standard.

- Macaulay constants $b_0, \dots, b_{N+1}$ of $d_{\mathcal{D}}$-standard dec. $\mathcal{D}$ of M:

$$b_i := \min\left\{d_{\mathcal{D}}, 1 + \deg \mathcal{D}_i\right\} = \begin{cases} d_{\mathcal{D}} & \text{if } \mathcal{D}_i = \emptyset \\ 1 + \deg \mathcal{D}_i & \text{otherwise.} \end{cases}$$

  where $\mathcal{D}_i := \left\{(w, y, h) \in \mathcal{D} : \#y \geq i\right\}$.
- It follows that $b_0 \geq \cdots \geq b_{N+1} = d_{\mathcal{D}}$.
- For $M = \mathrm{nf}_G(R)$, where G is a Gröbner basis of I, the Macaulay constants of all 0-standard decompositions are the same; for $d \geq b_0$:

$$H_{R/I}(d) = \binom{d - b_{N+1} + N}{N} - 1 - \sum_{i=1}^{N} \binom{d - b_i + i - 1}{i}.$$

Theorem

Suppose R and the generators $f_1, \dots, f_n$ of I are homogeneous of degree at most d. Then the elements of the reduced Gröbner basis G of I have degree at most

$$D(N-1,d) = 2\left(\frac{d^2}{2}+d\right)^{2^{N-2}}.$$

- $I = (f_1) \oplus \mathrm{nf}_{G_2}(R) f_2 \oplus \cdots \oplus \mathrm{nf}_{G_n}(R) f_n$, where G_i is a Gröbner basis of $\big((f_1, \dots, f_{i-1}) : f_i\big)$ for $i = 2, \dots, n$.
- Let b_i be the M. constants for a d-standard dec. of I.
- Let a_i be the M. constants (for a 0-standard dec.) of $\mathrm{nf}_G(R)$.
- Via a combinatorial argument, one may show $a_j + b_j \leq D(N-j,d)$ for $j = 1, \dots, N-2$. In particular, $a_1 + b_1 \leq D := D(N-1,d)$.
- Degrees of elements in G are bounded by b_0, but $\max\{a_0, b_0\} = \max\{a_1, b_1\} \leq D$.

Conclusion

A doubly exponential upper bound on Gröbner bases implies degree bounds on

- syzygies;
- normal forms;
- Gröbner bases of two-sided ideals.

Is there a single exponential bound in certain cases: e.g., holonomic ideals in the Weyl algebra?

- holonomic is an “analog” of 0-dimensional in the commutative case;
- no Bézout bound in the ring of differential operators with coefficients in rational functions; instead, weak Bézout bound, which is doubly exponential (Grigoriev);
- a single exponential bound is drawn for GKZ-hypergeometric ideals (SST).