Math 265 Linear Algebra **Sample Final** **Fall 2003**

1. It is given that $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 \\ -1 & 1 & 5 & -1 & -3 \\ 0 & 2 & 6 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$, $\mathbf{rref}\,(A) = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) Find the rank of A.

   (b) Find the nullity of A.

   (c) Find a basis for the column space of A.

   (d) Find a basis for the row space of A.

   (e) Find a basis for the null space of A.

   (f) Find a basis for the orthogonal complement of the row space of A.

2. Find an equation relating a, b and c so that the linear system
$$\begin{cases} 2x + 2y + 3z = a \\ 3x - y + 5z = b \\ x - 3y + 2z = c \end{cases}$$
is consistent for any values a, b and c which satisfy that equation.

3. Determine the values of a so that the linear system
$$\begin{cases} x + y + z = 2 \\ 2x + 3y + z = 5 \\ 2x + 3y + (a^2 - 3)z = a + 1 \end{cases}$$
has (a) no solution, (b) a unique solution, and (c) infinitely many solutions.

4. Find the value of y in the solution of the linear system
$$\begin{cases} 2x + 123y - z = 1 \\ x + 456y + z = 0 \\ 2x + 789y + z = 1 \end{cases}$$
if you know that the deteminant of the matrix of the above system is equal to -420.

5. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix},\ L = \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix},\ L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}.$$

Find the standard matrix for L.

6. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$L\left(\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \ L = \left(\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \ L\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}.$$

Find $L\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$.

7. Find the standard matrix of the linear transformation L defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - z \\ x + 2y + z \\ 3x - y \end{bmatrix}$$

8. If A is a 5×5 matrix and $\det A = 3$ find $\det(A^2 \cdot (\mathrm{adj}A)^2)$.

9. Find a basis for $S = \{t^2 + 1,\ t^2 + 2t,\ 3t^2 + t - 1\}$. Does $6t^2 - 1$ belong to span(S)?

10. For what values of d are the vectors $[1,\ 3,\ d]$, $[1,\ 1,\ 0]$ and $[0,\ 1,\ 1]$ linearly independent?

11. Compute adj(A) and A^{-1} for $A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 1 & 3 \\ 2 & -1 & 1 \end{bmatrix}$.

12. Find ALL diagonal matrices similar to $A = \begin{bmatrix} 1 & 1 \\ -5 & 7 \end{bmatrix}$.

13. Let us consider a triangle $\Delta(P, Q, R)$ in $\mathbb{R}^3$ whose vertices are as follows:

$$P = (-2\sqrt{3}, 1, -2),\ Q = (3\sqrt{3}, -3, 1),\ R = (-2\sqrt{3}, 7, 6)$$

(a) Find length of all sides of $\Delta(P, Q, R)$.

(b) Find all angles of $\Delta(P, Q, R)$.

14. Use the Gram-Schmidt process to construct an orthonormal basis for the subspace W of $\mathbb{R}^4$ spanned by

$$\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \end{bmatrix}\right\}.$$

15. Let W be the subspace of $\mathbb{R}^3$ spanned by the vector $w = [1\ 1\ -1]$.
   (a) Find a basis for the orthogonal complement $W^\perp$ of W.
   (b) Find an orthonormal basis for $W^\perp$.

16. Let W be the subspace of $\mathbb{R}^4$ spanned by the sset of vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 4 \end{bmatrix}$. (a) Find the projection of $\mathbf{v}$ onto W. (b) Find the distance from $\mathbf{v}$ to W.

17. For each of the following sets of vectors, determine if it is a vector (sub)space:
   (a) The set of all vectors in $\mathbb{R}^4$ with the property $2x_1 + x_2 - 3x_3 + x_4 = 0$; Yes No
   (b) The set of all vectors in $\mathbb{R}^4$ with the property $x_1^3 = x_2^3$, $x_3 = x_4$; Yes No
   (c) The set of all vectors in $\mathbb{R}^3$, which have the form $(0, a - b + c, 3b + c)$ where a, b and c are arbitrary real numbers; Yes No
   (d) The set of all polynomials P in the space of all polynomials of degree at most 7, with the property $P(2) = 0$; Yes No
   (e) The set of all nonsingular matrices in the space of 3×3 matrices; Yes No

18. Determine if each of the following sets of vectors is linearly independent or linearly dependent:
   (a) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$; Independent Dependent
   (b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \right\}$; Independent Dependent
   (c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \right\}$; Independent Dependent

(d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} \right\}$; Independent Dependent

(e) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Independent Dependent

19. We have a 3×3 matrix $A = \begin{bmatrix} a & 1 & 2 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$ with $\det(A) = 3$. Compute the determinant of the following matrix

(a) $\begin{bmatrix} a-2 & 1 & 2 \\ b-4 & 3 & 4 \\ c-6 & 5 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 7a & 7 & 14 \\ b & 3 & 4 \\ c & 5 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} a & 1 & 1 \\ b & 3 & 3 \\ c & 5 & 5 \end{bmatrix}$

(d) A^T

(e) $(5A)^{-1}$

20. Let $A = \begin{bmatrix} 3 & -5 \\ 1 & -3 \end{bmatrix}$.

(a) Find all eigenvalues of A.

(b) For each eigenvalue above, find an eigenvector of A associated to it.

(c) Find a diagonal matrix D and a nonsingular matrix P such that $P^{-1}AP = D$.

(d) Find A^{47}.

21. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(a) Find all eigenvalues of A.

(b) For each eigenvalue as above, find an eigenvector of A associated to it.

(c) Find a diagonal matrix D and an orthogonal matrix P such that $P^T AP = D$.

22. Determine an invertible matrix $\hat{A}$ and a vector $\hat{b}$ such that the solution to $\hat{A}\hat{X} = \hat{b}$ is

the least squares solution to $AX = b$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

23. Find the least squares fit line for the points

$$(1, 2), (2, 1), (3, 3), (4, 3).$$

24. Consider the homogeneous linear system of differential equations $\mathbf{x}'(t) = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & -1 \\ 8 & 8 \end{bmatrix}$.

   (a) Find all eigenvalues of A.

   (b) Find the general solution to the linear system of differential equations.

   (c) Solve the initial value problem for the given conditions $x_1(0) = 3$ and $x_2(0) = -10$.