Hints on how to solve some problems.
No warranties on any computations.

1. In the equation $x^2 \pm 5y^a \pm 21z^b = c$, we allow all possible choices of plus and minus, and for a, b being either 1 or 2, and for c being either 1 or 0. Making random choices (flip a coin) for all 5 parameters, sketch the surfaces to the equation in at least 4 cases.

2. Discuss the geometric/physical meaning of tangent, normal, binormal, curvature, torsion, kissing plane, kissing circle.

   - unit tangent: direction in which you are moving right now; lenght =1
   - normal: the amount of force that is changing your current direction (ignoring the change in direction of the motion, the usual acceleration)
   - bitangent: the unit verctor perpendicular to tangent and normal, chosen to make T, N, B a RHS. Also, the direction perpendicular to the plane of motion (spanned by T, N)
   - curvature: the rate of change over time by which your direction changes, rescaled by the speed at which you go (the rescaling is done so that you can compare the shapes of curves, irrespective of the speed at which you move though them)
   - torsion: the rate of change at which the direction of the binormal changes. It expresses changes of the plane of motion.
   - kissing plane: the plane spanned by T, N; also known as the plane of motion
   - the circle in the kissing plane that touches the point on the curve and has in the place of contact the same tangent direction and the same curvature.

3. Determine whether the line of intersection of $3x - y + 2z = 4$ and $2x + y - z = 1$ meets the plane $x - y - z = 3$. In general, given 3 planes, devise a quick test that checks whether the intersection of all three planes is empty or not. Be careful not to forget special cases.

   The line of intersection moves in direction $(3, -1, 2) \times (2, 1, -1)$ since these two vectors are the normals to the first two planes (and if you are perpendicular to the things that are perpendicular to a given plane, then you are in the plane). This product is $(-1, 7, 5)$ I think. That is a direction not equal to the normal $(1, -1, -1)$ of the third plane, and so any line with slope $(-1, 7, 5)$ must pass through the third plane.

   The general test is: if the box product of the three normals is nonzero, the planes will meet in one point exactly. The other cases can be of various sorts, depending on how parallelity works out.

4. Consider the curve $(x, y, z) = (t, t^2, t)$. Compute T, N, B. Where does this curve have maximal curvature, and where maximal torsion?

   $v = (1, 2t, 1)$ and so $T = (1, 2t, 1)/\sqrt{2 + 4t^2}$. Then $dT/dt = ((0, 2, 0)\sqrt{2 + 4t^2} - (1, 2t, 1) \cdot \sqrt{2 + 4t^2}^{-1} \cdot 4t)/(2 + 4t^2) = ((0, 2, 0)(2 + 4t^2) - (1, 2t, 1) \cdot 4t)/\sqrt{2 + 4t^2}^3 = (-4t, 4, -4t)/\sqrt{2 + 4t^2}^3$.

Next, scale dT/dt by $1/||dT/dt||$, to get N. Then compute $B = T \times N$. Then compute $(dB/dt)/||v||$.

Since $\kappa = ||dT/dt||/||v||$, take this expression and ask for what t it is biggest.

Since $\tau = ||dB/dt||$, take that expersion and see where it is biggest.

5. Find the point P on the cone $4x^2 = y^2 + z^2$ that is closest to $Q = (1, 2, 3)$. (Hint: first, rotate so that Q lies in the xz-plane, call this new point Q'. Let P' be the rotated optimal point P. Then use that the line from P to Q is perpendicular to the surface to find a linear equation that P' must satisfy. Use it and the cone equation to find a simple quadratic relation that the x and z coordinate of P' must satisfy. Factor it. You get 2 possible linear relations between x and z for the optimal P'. In each case we now have 2 linear equations for P. That is, you found two lines, on one of which P' must sit. Now find the best P' for each line with known methods. Finally, rotate back to get P.)

   The cone is a cone over a circle. Its symmetry axis is the x-axis. So the point on the cone closest to P is on a line from P to some point on the x-axis. Moreover, this line meets the cone at right angles.

   Let us first rotate the whole picture about the x-axis, so as to make the point P lie on the plane $y = 0$. This produces the point $P' = (1, 0, \sqrt{2^2 + 3^2})$.

   Slice the cone with the xz-plane. The cone intersected with that plane has equation $4x^2 = z^2$, so consists of the lines $2x = z$ and $2x = -z$. The point P' is closer to the former. The straight line L' that links P' to the line $2x + 0y - z$ in the plane $y = 0$ has direction $(2, 0, 1)$. So, look at where $P' + t(2, 1)$ meets $2x = z$. This happens when $t = (\sqrt{13} - 2)/3$. This now tells you where L' meets $2x = z$ by plugging in. Call that point P", a point on the xz-plane, and on the line $2x = z$. It will look like $(a, 0, c)$. (You need to compute a, c).

   Now rotate this point back, so that P' is back at P. The rotation that carries $(1, 2, 3)$ to $(1, 0, \sqrt{13})$ is a rotation by $\arctan(3/2)$ about the x-axis. So we rotate by $\arctan(-3/2)$ the point $(a, 0, c)$. You get $(a, 2c/\sqrt{13}, 3c/\sqrt{13})$.

   Again, no warranties.

6. Find the angle between the sides of a regular tetrahedron. (The angle between the *edges* is obviously 60 degrees, so that is not what I am asking).

   The vertices of the tetrahedron can be chosen as $(0, 0, 0), (1, 0, 0), (1/2, \sqrt{3}/2, 0), (1/2, 1/3, \sqrt{23}/6)$. The normal vector to the side with the first three on it is $(1, 0, 0) \times (1, \sqrt{3}, 0) = (0, 0, 1)$ obviously, and the one normal to the side with the first second and fourth is $(1, 0, 0) \times (3, 2, \sqrt{23}) = (0, -\sqrt{23}, 2)$. Now use the cosine formula on these.

7. Find the surface area of the tetrahedron with vertices $(t, t{\cdot}t^3)$ for $t = 0, 1, 2, 3$.

   Take any three vertices with coordinates as found above, take 1/2 of the cross product of the difference vectors. (You need to do this in all 4 possible cases of choosing 3 vertices; then add the results. Watch out for sign issues.)

8. A boat with broken rudder is drifting down the Mississippi, hoping to get to the port in Natchez. At noon, the boat is 2 km North of the port, and 0.3 km West. The current leads it straight south at 13 m per second. The boat also has a sail up. There is a gentle wind blowing from the Northwest, at 5 m per second. The sail catches 60 percent of the wind speed. In order to make it to the port, should the people on the boat row forward or backward? And at what speed relative to the water?

   Write down the vector that they are aiming for. Write down the river current vector, and the wind vector, and the rowing vector (the latter should be of the sort $(0, r)$ where r is the rowing speed in the North direction). Add the last 3 and equate its direction with the direction of the first. That will tell you r.

9. Let $f(x, y, z)$ describe a surface in 3-space that contains the point P. Explain why one should not always hope for a "kissing sphere" to S at P, although curves always have a kissing circle. (The best way is to check through some basic examples of quadric surfaces and see whether such kissing sphere would make sense). If you can find such example, devise a test which determines whether such kissing sphere exists.

   Look at a cylinder. In one sense the sphere should be of the radius of the cylinder, in the other it should be of infinite radius.

10. Suppose S is a quadric surface in 3-space. Suppose also that L is any line that is not completely inside S. How many points can the intersection $S \cap L$ have? For each possible answer, describe one example.

    If you plug a linear equation into a quadric, eliminating y leads to a degree 2 polynomial in x. So, you should expect 2 solutions. There might be fewer, since the line could be tangent, or miss altogether. (Even with complex coefficients, $x^2 + y^2 = 1$ does not meet the z-axis. But it wants to "meet it" in a tangential way at infinity...)).

11. A ball is thrown (at time t=0) at an angle of 30 degrees upwards against the horizontal, with speed 32 m/s. Assuming a level playing field, describe location and speed in terms of t. Compute total travel distance. Find another angle that makes the ball land in the same spot, assuming the same initial speed.

    The ball flies along $(x', y') = (32\cos(30), 32\sin(t) - gt)$. So $(x, y) = (32\cos(30)t + c_1, 32\sin(30)t - gt^2/2 + c_2)$. We know $c_1 = 0 = c_2$ from initial conditions. So the next time $y = 0$ is when $t = 64\sin(30)/g$. And at that time, $x = 32\cos(30)64\sin(30)/g$.

    If you want to reach that flight distance with another angle α you need $32\cos(30)64\sin(t)/g = 32\cos(\alpha)64\sin(\alpha)/g$ which means you want that $\sin(2 \cdot 30) = \sin(2\alpha)$. That works for $\alpha = 30, 60$. So, the other angle is 60.