MA 511, Session 1

Linear Systems

A basic problem in linear algebra is that of solving the linear system

$$(1) \qquad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_2 \\ \dots\dots\dots\dots & \dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m \end{cases}$$

This is a system of m **linear equations** in n unknowns. The a_{ij}'s and the b_j's are constants called, respectively, the coefficients of the system and the right-hand sides (or forcing terms). If $b_j = 0$ for $1 \leq j \leq m$, the system is homogeneous.

The locus of points satisfying the equation

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

is an $(n-1)$-dimensional plane in the n-dimensional space $\mathbb{R}^n$.

Example: For $m = n = 2$, a system

$$\begin{cases} x + y &= 1 \\ -x + y &= 0 \end{cases}$$

has a solution $(x, y) = (\frac{1}{2}, \frac{1}{2})$, which is the intersection point of the lines $x + y = 1$ and $x = y$.

Example: For $m = n = 3$, a system

$$\begin{cases} 2x - y &= 0 \\ -x + 2y - z &= 1 \\ -y + 2z &= 2 \end{cases}$$

has a solution $(x, y, z) = (1, 2, 2)$ which is the intersection point of the planes $2x - y = 0$, $-x + 2y - z = 1$ and $-2y + z = 2$.

In <u>parametric form</u> a line can be described by one parameter (in a space of any number of dimensions n)

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = t \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{pmatrix},$$

where $\overrightarrow{c} = (c_1, c_2, \ldots, c_n)^T$ is a vector in the direction of the line and $\overrightarrow{d} = (d_1, d_2, \ldots, d_n)^T$ is a point on the line.

Similarly, a plane can be described by two parameters (in a space of any number of dimensions n)

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = s \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{pmatrix} + t \begin{pmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{pmatrix}.$$

Here $\overrightarrow{c} = (c_1, c_2, \ldots, c_n)^T$ and $\overrightarrow{d} = (d_1, d_2, \ldots, d_n)^T$ are vectors in the plane with different directions, and $\overrightarrow{e} = (e_1, e_2, \ldots, e_n)^T$ is a point on the plane.

Let us introduce the following notation for the *columns* of the array (a_{ij}) and for the right-hand sides in (1):

$$\vec{c}_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \cdot \\ \cdot \\ \cdot \\ a_{mi} \end{pmatrix}, \quad 1 \leq i \leq n, \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix}.$$

Then, the system (1) can be re-written as

$$x_1 \vec{c}_1 + x_2 \vec{c}_2 + \cdots + x_n \vec{c}_n = \vec{b}.$$

We immediately see that the system is consistent (i.e., it admits solution) if and only if $\vec{b}$ is a combination of the columns of the array (a_{ij}) (*algebraic approach*).

Looking at the *rows* of (1), we see the system is consistent if, and only if the m planes intersect (*geometric approach*).

Gaussian Elimination

Example: Consider in $\mathbb{R}^3$ the three planes given by

$$\begin{cases} x + 2y + 3z &= 1 \\ 4x + 5y + 6z &= 4 \\ 7x + 8y + 9z &= 8 \end{cases}$$

Then,

$$\begin{cases} x + 2y + 3z &= 1 \\ 0x - 3y - 6z &= 0 \\ 0x - 6y - 12z &= 1 \end{cases}$$

and

$$\begin{cases} x + 2y + 3z = 1 \\ 0x + y + 2z = 0 \\ 0x + y + 2z = -\frac{1}{6} \end{cases}$$

We see the system is inconsistent since the second and third equations cannot be satisfied simultaneously.

If we change the 8 on the right of the third equation to 7, then we have

$$\begin{cases} x + 2y + 3z &= 1 \\ 4x + 5y + 6z &= 4 \\ 7x + 8y + 9z &= 7 \end{cases}$$

or

$$\begin{cases} x + 2y + 3z &= 1 \\ 0x - 3y - 6z &= 0 \\ 0x - 6y - 12z &= 0 \end{cases}$$

and

$$\begin{cases} x + 2y + 3z = 1 \\ 0x + y + 2z = 0 \\ 0x + y + 2z = 0 \end{cases}$$

so that

$$\begin{cases} x + 2y + 3z = 1 \\ 0x + y + 2z = 0 \end{cases}$$

We now see the locus of solutions consists of the line

$$z = t, \quad y = -2t, \quad x = 1 + t.$$

Similarly, if the 9 in the original system were changed to any other number r, then the intersection of the three planes would be a single point:

$$z = \frac{1}{r-9}, \quad y = -\frac{2}{r-9}, \quad x = \frac{r-8}{r-9}.$$

Gaussian elimination provides a systematic way of doing such analysis.

This involves two distinct parts: *elimination* by row operations, and *back substitution*.

Three types of elementary row operations are permitted that produce *equivalent* systems (i.e. systems with exactly the same solutions):

- Interchanging two rows
- Multiplying a row by a non-zero number
- Replacing a row by itself plus a multiple of another row