MA 511, Session 2

Matrices

A rectangular array of numbers with m horizontal lines and n vertical lines is called a $m \times n$ matrix. The horizontal lines of the array are called the rows and the vertical ones the columns.

Notation:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

a_{ij} = coefficient in the i-th row and j-th column.

Matrices with $m = 1$ or $n = 1$ are called, respectively, row vectors and column vectors:

$$(a_1, a_2, \dots, a_n) \qquad \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{pmatrix}$$

Addition: component-wise (matrices must be of the same dimensions)

Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 2 \\ -2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 5 \\ 2 & 5 & 9 \end{pmatrix}$$

Multiplication by Scalar: component-wise

Example:

$$2 \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$$

Inner Product (dot product) of Vectors: componentwise, then add

Example:

$$(a_1, a_2, \ldots, a_n) \cdot \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Multiplication of $m \times n$ matrix times n-component column vector: done row-by-row as a dot product

$$\begin{pmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ \cdot \\ R_m \end{pmatrix} w = \begin{pmatrix} R_1 \cdot w \\ R_2 \cdot w \\ \cdot \\ \cdot \\ \cdot \\ R_m \cdot w \end{pmatrix}$$

$R_1, \ldots, R_m$ are n-component row vectors

w is a n-component column vector

Example: "row-at-a-time"

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1(1) + 2(0) + 3(-1) \\ 4(1) + 5(0) + 6(-1) \\ 7(1) + 8(0) + 9(-1) \\ 2(1) + 1(0) + 0(-1) \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \\ 2 \end{pmatrix}$$

"Column-at-a-time" Multiplication:

$$(C_1\ C_2\ \dots\ C_n)\begin{pmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_n \end{pmatrix} = w_1 C_1 + w_2 C_2 + \cdots + w_n C_n$$

$C_1, \dots, C_n$ are m-component column vectors

$w_1, \dots, w_n$ are the components of column vector w

In the previous example:

$$1\begin{pmatrix} 1 \\ 4 \\ 7 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 5 \\ 8 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 3 \\ 6 \\ 9 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \\ 2 \end{pmatrix}$$

Multiplication of $m \times n$ matrix times $n \times p$ matrix: done row-by-row times column-by-column as a dot product

$$\begin{pmatrix} R_1 \\ R_2 \\ . \\ . \\ . \\ R_m \end{pmatrix} (C_1 \; C_2 \; \dots \; C_p) =$$

$$\begin{pmatrix} R_1 \cdot C_1 & R_1 \cdot C_2 & \dots & R_1 \cdot C_p \\ R_2 \cdot C_1 & R_2 \cdot C_2 & \dots & R_2 \cdot C_p \\ \dots & \dots & \dots & \dots \\ R_m \cdot C_1 & R_m \cdot C_2 & \dots & R_m \cdot C_p \end{pmatrix}$$

$R_1, \dots, R_m$ are n-component row vectors

$C_1, \dots, C_p$ are n-component column vectors

Example: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 4 & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 10 & -4 \\ 22 & -4 \end{pmatrix}$

Formal Rule:

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{1p} \\ b_{21} & \dots & b_{2p} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1p} \\ c_{21} & \dots & c_{2p} \\ \dots & \dots & \dots \\ c_{m1} & \dots & c_{mp} \end{pmatrix}$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Properties of Matrix Multiplication: If the dimensions are appropriate so that the operations are defined,

$$\begin{array}{lrcl} \text{(i)} & (AB)C & = & A(BC) \\ \text{(ii)} & A(B+C) & = & AB+AC \\ \text{(iii)} & (A+B)C & = & AC+BC \end{array}$$

<u>**Usually**</u> $AB \neq BA$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$$

Be careful with the dimensions when multiplying matrices.

Example:

$$(1\ 2\ 3) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 2$$

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} (1\ 2\ 3) = \begin{pmatrix} -1 & -2 & -3 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

Definition: the $n \times n$ identity matrix is $(\mathbb{I}_n)_{ij} = \delta_{ij}$ ($\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$ is the Kronecker symbol.)

$$\mathbb{I} = \mathbb{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

It functions like the number 1: If A is a $m \times n$ matrix, then

$$A\mathbb{I}_n = \mathbb{I}_m A = A$$

Elementary Matrices (3×3 case):

1. P_{ij} obtained by interchanging the i-th and j-th rows in $\mathbb{I}$

$$\underline{\text{Ex}}: \quad P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. $E_i(c)$ obtained by multiplying i-th row by nonzero number c in $\mathbb{I}$

$$\underline{\text{Ex}}: \quad E_2(4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. $E_{ij}(c)$ matrix formed by replacing i-th row by itself plus c times the j-th row in $\mathbb{I}$

$$\underline{\text{Ex}}: \quad E_{21}(4) = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Each row operation in the Gaussian elimination process on an arbitrary matrix can be accomplished by multiplying the matrix on the left by the corresponding elementary matrix:

Example: Multiplication on the left by $E_{21}(4)$ replaces the second row by itself plus 4 times the first row

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 2 \\ 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 2 \\ -3 & 2 & 3 & 12 \\ 6 & 7 & 8 & 9 \end{pmatrix}$$

Multliplication on the left by P_{23} interchanges the second and third rows of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 5 & 5 & 5 \\ 6 & 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 5 & 5 & 5 & 5 \end{pmatrix}$$