MA 511, Session 3

LU Decomposition

Remark: It is easy to "undo" what an elementary matrix does.

$$\begin{aligned} P_{ij}P_{ij} &= \mathbb{I} \\ E_i(c)E_i(\tfrac{1}{c}) &= \mathbb{I} \\ E_{ij}(c)E_{ij}(-c) &= \mathbb{I} \end{aligned}$$

Go back to the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{cases}$$

It can be written using matrices as

$$Ax = b$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{pmatrix}.$$

Consider now Gaussian elimination applied to a (square) $n \times n$ matrix A.

By elementary row operations, A can be brought to <u>upper triangular form</u>

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & \bar{a}_{22} & \dots & \bar{a}_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \bar{a}_{nn} \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

Using elementary matrices:

$$E_{32}(-2)E_{31}(-7)E_{21}(-4) \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = E_{21}(4)E_{31}(7)E_{32}(2) \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

Remark: In our algorithm, we began with the top row and worked our way down. This implies that the elementary matrices $E_{32}(-2), E_{31}(-7), E_{21}(-4)$ (and hence also $E_{21}(4), E_{31}(7), E_{32}(2)$) are obtained from $\mathbb{I}$ introducing no changes above the diagonal. Therefore, their product, just like themselves, is lower triangular:

$$L = E_{21}(4)E_{31}(7)E_{32}(2)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix}$$

Note that the entries l_{21}, l_{31}, l_{32} of L below the diagonal are exactly the multipliers 4, 7, and 2 in the elementary matrices $E_{21}(4)$, $E_{31}(7)$, $E_{32}(2)$ above.

Therefore, <u>if no row exchanges are needed</u>, we may write

$$A = LU$$

where L is lower triangular with 1's on the diagonal and U is upper triangular. This is the LU decomposition of A.

From here we can derive a more symmetric factorization of A. Suppose none of the coefficients on the diagonal of U are zero. Then,

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ l_{n1} & l_{n2} & \dots & 1 \end{pmatrix} \begin{pmatrix} d_1 & u_{12} & \dots & u_{1n} \\ 0 & d_2 & \dots & u_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ l_{n1} & l_{n2} & \dots & 1 \end{pmatrix}$$

$$\begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_n \end{pmatrix} \begin{pmatrix} 1 & u_{12}/d_1 & \dots & u_{1n}/d_1 \\ 0 & 1 & \dots & u_{2n}/d_2 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

This is the LDU decomposition of A.

Suppose now that we do need row exchanges. These are accomplished multiplying by P_{ij} matrices. A product P of such elementary matrices is called a permutation matrix.

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

We can do the necessary row exchanges first, then the LU factorization:

$$P_{23}A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & 8 \\ 1 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

Showing the elimination steps:

$$E_{31}(-1)E_{21}(-2)P_{23}A = U$$

Now $PA = LU$, where $P = P_{23}$, $L = E_{21}(2)E_{31}(1)$.

This can always be done since, for $i, k, l > j$, $P_{kl}E_{ij}(c)$ equals either $E_{lj}(c)P_{kl}$, when $k = i$, or $E_{kj}(c)P_{kl}$, when $l = i$, or else $E_{ij}(c)P_{kl}$.

Definition: We say A is nonsingular if there is a sequence of row operations on A that yields U with nonzero coefficients on the diagonal. Otherwise A is singular.

Suppose A is nonsingular. Then, there is a sequence of row operations taking A to U. Since the coefficients on the diagonal of U are nonzero, we can use them as pivots to make every coefficient above the diagonal of U become zero using more elementary row operations (multiplication from the left by upper triangular elementary matrices). Finally, dividing each row by the coefficient on the diagonal (pivot), we can make all the coefficients on the diagonal equal to 1.

In summary: If A is nonsingular, there is a sequence of elementary row operations that take A to the identity matrix.