MA 511, Session 4

Inverses and Transposes

Definition: If A is a $m \times n$ matrix and a $n \times m$ matrix B exists such that $BA = \mathbb{I}_n$, then B is called a left inverse of A. Similarly, if a $n \times m$ matrix C exists such that $AC = \mathbb{I}_m$, then C is called a right inverse of A.

It is easy to see that if A has both a right and a left inverse, they must be equal: $BA = \mathbb{I}_n$ and $AC = \mathbb{I}_m$ imply

$$B = B\mathbb{I}_m = B(AC) = (BA)C = \mathbb{I}_n C = C.$$

Fact: (nontrivial) If A, B are $n \times n$ matrices and $AB = \mathbb{I}$, then also $BA = \mathbb{I}$.

Notation: The inverse of A is denoted by $B = A^{-1}$.

Note that $AA^{-1} = A^{-1}A = \mathbb{I}$.

Definition: A $n \times n$ matrix A is <u>invertible</u> if there exists a $n \times n$ matrix B such that $BA = \mathbb{I}$.

We have seen that, if A is nonsingular, there is a sequence of elementary row operations on A that yields the identity matrix. This means that there exists a matrix E (the product of the elementary matrices that perform the necessary elementary row operations) such that

$$EA = \mathbb{I}.$$

This means that $E = A^{-1}$.

Fact: Every nonsingular matrix is invertible.

Next, let us write E as a product of elementary matrices:

$$E = E_1 \dots E_k.$$

Then, $E_1 \dots E_k A = \mathbb{I}$ gives $A = E_k^{-1} \dots E_1^{-1}$.

Fact: Every nonsingular matrix is a product of elementary matrices (since the inverse of an elementary matrix is itself an elementary matrix).

Also, $E_1 \dots E_k A = \mathbb{I}$ gives

$$A^{-1} = E_1 \dots E_k = E_1 \dots E_k \mathbb{I}.$$

This provides an explicit way for computing the inverse of A.

Fact: Suppose A is nonsingular. If we juxtapose $\mathbb{I}$ to A (forming a $n \times 2n$ matrix) and then apply Gaussian elimination until the left half (originally, A) of the augmented matrix becomes the identity $\mathbb{I}$, then the right half (originally, $\mathbb{I}$) becomes A^{-1}.

$$[A|\,\mathbb{I}\,] \longrightarrow [\,\mathbb{I}\,|A^{-1}]$$

Example:

$$\begin{pmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 4 & 5 & 6 & | & 0 & 1 & 0 \\ 7 & 8 & 8 & | & 0 & 0 & 1 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 0 & 0 & | & -\frac{8}{3} & \frac{8}{3} & -1 \\ 0 & 1 & 0 & | & \frac{10}{3} & -\frac{13}{3} & 2 \\ 0 & 0 & 1 & | & -1 & 2 & -1 \end{pmatrix}$$

So,

$$A^{-1} = \begin{pmatrix} -\frac{8}{3} & \frac{8}{3} & -1 \\ \frac{10}{3} & -\frac{13}{3} & 2 \\ -1 & 2 & -1 \end{pmatrix}.$$

Fact: (nontrivial) Every invertible matrix is nonsingular. Hence, a square matrix is nonsingular if, and only if it is invertible.

<u>Properties of Inverses</u>:

$$(AB)^{-1} = B^{-1}A^{-1}, \qquad\qquad (A^{-1})^{-1} = A,$$

since

$$\begin{aligned}(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B& \\ = B^{-1}\mathbb{I}B = B^{-1}B = \mathbb{I}&\end{aligned}$$

and $A^{-1}A = \mathbb{I}$.

Transposes:

The <u>transpose</u> of a $m \times n$ matrix A is the $n \times m$ matrix A^T, where $(A^T)_{ij} = (A)_{ji}$.

Example: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Definition: A matrix A is <u>symmetric</u> if $A^T = A$.

Note that a symmetric matrix must be square.

Properties of Transposition:

- $(AB)^T = B^T A^T$
- If A is invertible, $(A^{-1})^T = (A^T)^{-1}$
- $(A^T)^T = A$