MA 511, Session 5

Review

1. Show that the product of upper triangular matrices is upper triangular.

Let

$$S = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ 0 & s_{22} & \dots & s_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & s_{nn} \end{pmatrix}, \quad T = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ 0 & t_{22} & \dots & t_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & t_{nn} \end{pmatrix}.$$

Note that upper triangular means $s_{ij} = t_{ij} = 0$ for $0 \le j < i \le n$. Let now

$$ST = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix}$$

where

$$c_{ij} = \sum_{k=1}^{n} s_{ik} t_{kj}.$$

We need to show that, if $0 \le j < i \le n$, then $c_{ij} = 0$. Let $j < i$:

(i) Suppose $s_{ik} \neq 0$. Then, $i \le k$ and so $j < k$, implying $t_{kj} = 0$.

(ii) Similarly, if $t_{kj} \neq 0$, then $k \le j$ and so $k < i$, implying that $s_{ik} = 0$.

Therefore, if $j < i$ it follows that $s_{ik}t_{kj} = 0$ for $1 \le k \le n$, and thus $c_{ij} = 0$ whenever $j < i$. Hence, ST is upper triangular.

2. Solve by elimination and back substitution the system

$$\begin{cases} u \qquad + w = 4 \\ u + v \qquad = 3 \\ u + v + w \ = 6 \end{cases}$$

We have

$$\begin{cases} u \qquad + w = 4 \\ \qquad v - w = \ -1 \\ \qquad v \qquad = 2 \end{cases}$$

and

$$\begin{cases} u \qquad + w = 4 \\ \qquad v - w = \ -1 \\ \qquad\quad w \quad = 3 \end{cases}$$

leading to $w = 3$, $v = -1 + w = 2$, and $u = 4 - w = 1$.

3. Describe geometrically the locus of solutions of the system

$$\begin{cases} u + v + w + z = 4 \\ 2u - v + 2w + z = 6 \end{cases}$$

We use Gaussian elimination to arrive at the system

$$\begin{cases} u + v + w + z = 4 \\ \quad -3v \quad\quad - z = -2 \end{cases}$$

with four unknowns, but just two pivots (1 and -3, respectively the coefficients of u in the first equation and of v in the second). There are 2 parameters,

$$z = t, \qquad w = s,$$

and then back substitution gives

$$v = \tfrac{2-z}{3} = -\tfrac{t}{3} + \tfrac{2}{3},$$
$$u = 4 - v - w - z = \tfrac{t}{3} - \tfrac{2}{3} - s - t + 4 = -\tfrac{2}{3}t - s + \tfrac{10}{3}.$$

Finally,

$$\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = s \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{10}{3} \\ \frac{2}{3} \\ 0 \\ 0 \end{pmatrix}$$

This is a 2-dimensional plane in $\mathbb{R}^4$.

4. If E is an elementary matrix, is E^T also an elementary matrix?

Yes, as we can see by looking at the three types of elementary matrices:

$$\begin{aligned} &\text{(i)} & P_{ij}^T &= P_{ij}, \\ &\text{(ii)} & E_i(c)^T &= E_i(c), \\ &\text{(iii)} & E_{ij}(c)^T &= E_{ji}(c). \end{aligned}$$

If A is nonsingular, is A^T nonsingular?

Yes, since $(A^T)^{-1} = (A^{-1})^T$.

5. Let us consider a counterclockwise rotation of the plane about the origin by an angle α.

Take a point $\begin{pmatrix} x \\ y \end{pmatrix}$ and let

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

That is $\begin{cases} x' = x\cos\alpha - y\sin\alpha \\ y' = x\sin\alpha + y\cos\alpha \end{cases}$

In polar coordinates we write $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$. Then,

$$x' = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha = r\cos(\theta+\alpha)$$
$$y' = r\cos\theta\sin\alpha + r\sin\theta\cos\alpha = r\sin(\theta+\alpha)$$

Hence, $\begin{pmatrix} x' \\ y' \end{pmatrix}$ is $\begin{pmatrix} x \\ y \end{pmatrix}$ rotated through an angle α.

6. Find the inverse of $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

We use the Gauss-Jordan method.

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 2 & 0 & -\frac{2}{3} & \frac{4}{3} & \frac{2}{3} & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 2 & 0 & -\frac{2}{3} & \frac{4}{3} & \frac{2}{3} & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right)$$

Finally,

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}.$$

7. True or False?

(i) If a $n \times n$ matrix A has all the coefficients on its diagonal equal to zero, then A is singular.

False, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not singular.

(ii) If a $n \times n$ matrix A has two identical rows, then it is singular.

True, since Gaussian elimination produces a row full of zeros, which makes it impossible for A to have n pivots.

(iii) If $A^2 + A = \mathbb{I}$, then $A^{-1} = A + \mathbb{I}$.

True, since then $(A + \mathbb{I})A = A^2 + A = \mathbb{I}$.

(iv) $(\mathbb{I} - A^2) = (\mathbb{I} + A)(\mathbb{I} - A)$.

True, since

$$(\mathbb{I} + A)(\mathbb{I} - A) = \mathbb{I} - A + A - A^2 = \mathbb{I} - A^2.$$