MA 511, Session 6

Vector Spaces

One of the main features of the set of n-component column vectors $\mathbb{R}^n$ is that there are two algebraic operations on the set, addition of vectors and scalar multiplication of a number by a vector:

If $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$ (c is a scalar), then

$$u + v \in \mathbb{R}^n$$
$$cu \in \mathbb{R}^n$$

There are other familiar sets of mathematical objects that have the same feature.

Example: $V = \mathcal{C}[a,b]$, the set of continuous functions on the interval $[a,b]$. With the standard addition of functions and scalar multiplication of a function we have, for any $f, g \in \mathcal{C}[a,b]$ and any $c \in \mathbb{R}$,

$$f + g \in \mathcal{C}[a,b]$$
$$cf \in \mathcal{C}[a,b]$$

since sums and multiples of continuous functions are continuous.

Example: For fixed positive integers m and n, let $V = \mathcal{M}_{m\times n}$ be the set of $m \times n$ matrices with real coefficients.With the standard addition of matrices and scalar multiplication of a matrix we have, for any $A, B \in \mathcal{M}_{m\times n}$ and any $c \in \mathbb{R}$,

$$A + B \in \mathcal{M}_{m\times n}$$
$$cA \in \mathcal{M}_{m\times n}$$

Definition: An abstract real vector space is a set V on which an operation of addition and another of multiplication by scalars (real numbers) are defined so that, for any $u, v, w \in V$ and any $c, d \in \mathbb{R}$,

$$(*) \qquad \begin{aligned} u + v &\in \mathbb{V} \\ cu &\in \mathbb{V} \end{aligned}$$

and these operations satisfy the following eight properties:

1. $u + (v + w) = (u + v) + w$

   (associative property of vector addition)
2. $u + v = v + u$

   (commutative property of vector addition)
3. There is a "zero" vector 0 such that $u + 0 = 0$
4. For each $u \in V$ there is an additive inverse, $-u$, such that $u + (-u) = 0$
5. $1u = u, \quad 0u = 0,$
6. $(cd)u = c(du)$
7. $c(u + v) = cu + cv$
8. $(c + d)u = cu + du$

Definition: If W is a subset of a real vector space V (i.e. $W \subset V$) and (*) holds for W, then W is also a real vector space called a <u>subspace</u> of V.

Examples of Subspaces:

(i) W, the set of all vectors in $\mathbb{R}^3$ of the form $\begin{pmatrix} x \\ y \\ x+y \end{pmatrix}$ is a subspace of $\mathbb{R}^3$

(ii) W, the set of all vectors in $\mathbb{R}^3$ of the form $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ is a subspace of $\mathbb{R}^3$

(iii) W, the set of all vectors in $\mathbb{R}^3$ of the form $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ is **not** a subspace of $\mathbb{R}^3$

(iv) For any fixed $k \in \mathbb{N}$ (a natural number), $\mathcal{C}^k[a,b]$, the set of all functions with k continuous derivatives on $[a,b]$ is a subspace of $\mathcal{C}[a,b]$

In fact, we have $\cdots \subset \mathcal{C}^2[a,b] \subset \mathcal{C}^1[a,b] \subset \mathcal{C}[a,b]$

Example: Consider on the interval $[-1, 1]$ the functions

$$f(x) = |x|,$$
$$g(x) = x^{4/3}.$$

Then, $f \in \mathcal{C}[-1, 1]$ but $f \notin \mathcal{C}^1[-1, 1]$. Similarly, $g \in \mathcal{C}[-1, 1]$ and $g \in \mathcal{C}^1[-1, 1]$ but $g \notin \mathcal{C}^2[-1, 1]$.

(v) $\mathcal{P}_n$, the set of polynomials of degree less that or equal to n is a subspace of $\mathcal{P}$, the space of all polynomials. ($\mathcal{P}_0$ is the space of constant functions.)

In fact, we have

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_n \subset \mathcal{P}_{n+1} \subset \ldots$$

(vi) Let $A \in \mathcal{M}_{m \times n}$. W, the set of all solutions to the homogeneous system $Ax = 0$ is a subspace of $\mathbb{R}^n$, called the <u>null space</u> of A and denoted as $\mathcal{N}(A)$.

<u>Check:</u> Let $u, v \in W$ and let $c \in \mathbb{R}$. Then,

$$A(u + v) = Au + Av = 0 + 0 = 0$$
$$A(cu) = cAu = c0 = 0$$

The elements of a vector space are called <u>vectors</u>. In this context, when we consider the vector space $V = \mathcal{M}_{m\times n}$ of $m \times n$ matrices with real coefficients, each such matrix is a *vector* in V.

Definition: let $v_1, \ldots, v_k$ be vectors in the vector space V and let $c_1, \ldots, c_k \in \mathbb{R}$ be scalars. A <u>linear combination of $v_1, \ldots, v_k$</u> is any sum of multiples of them,

$$c_1 v_1 + \cdots + c_k v_k$$

Using (*) repeatedly we see that linear combinations of any vectors in a vector space V are always in V.

Definition: Given a $m\times n$ matrix A, its <u>column space</u>, $\mathcal{C}(a)$, is the set of all linear combinations of the columns of A (a subset of $\mathbb{R}^n$).

<u>Fact</u>: $\mathcal{C}(A)$ is a vector space, i.e. a subspace of $\mathbb{R}^n$.

With this new concept, we can state the condition for a linear system to be consistent as follows:

$$Ax = b \text{ has a solution } \iff b \in \mathcal{C}(A)$$

Indeed, $Ax = b$ means $b = x_1\vec{C}_1 + x_2\vec{C}_2 + \cdots + x_n\vec{C}_n$ where $\vec{C}_j$ is the j-th column of A.

Example: Determine the null space and the column space of

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

The LU factorization of A is

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}.$$

We immediately see that $Ax = b$ has a solution for *every* $b \in \mathbb{R}^2$, which means that $\mathcal{C}(A) = \mathbb{R}^2$. Similarly, $Ax = 0$ has $0 \in \mathbb{R}^2$ as unique solution, and thus $\mathcal{N}(A) = \{0\}$.