MA 511, Session 7

<u>The Solution of $Ax = b$</u>

Consider first the linear system

$$(1) \qquad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_2 \\ \cdots\cdots\cdots\cdots & \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = b_n \end{cases}$$

This is a (square) system of n **<u>linear equations</u>** in n unknowns, written in matrix form as

$$Ax = b.$$

When $b = 0$ the system is called <u>homogeneous</u>, and it may have a unique solution ($x = 0 \in \mathbb{R}^n$, the trivial one) or infinitely many.

The set W of all solutions to the homogeneous system is a subspace of $V = \mathbb{R}^n$:

For any $u, v \in V$ and any $c \subset \mathbb{R}$,

$$A(u+v) = Au + Av = 0 + 0 = 0$$
$$A(cu) = cAu = c0 = 0$$

Therefore, when the homogeneous system has a non-zero solution, u say, then all multiples of u are also solutions and are all different; thus infinitely many solutions do exist.

In the non-homogeneous case the set of all solutions is <u>not</u> a vector space. It may be the empty set, it may have exactly one vector, or infinitely many.

To see this, first observe that $Ax = b$ may have no solution (which is equivalent to saying $b \notin \mathcal{C}(A)$), leading to the first possibility. If the system does have a solution, x say, we notice that for any other solution of $Ax = b$, y say, the difference $x - y$ is a solution of the corresponding homogeneous system, i.e. $x - y = v \in \mathcal{N}(A)$. In fact,

$$A(x - y) = Ax - Ay = b - b = 0.$$

This means that any solution of the non-homogeneous system may be written in the form

$$y = x + v$$

for some $v \in \mathcal{N}(A)$. If $\mathcal{N}(A) = \{0\}$, then x is the only solution of the non-homogeneous system $Ax = b$; if $\mathcal{N}(A)$ has infinitely many vectors, then so does the solution set of the non-homogeneous system.

Remark: When A is nonsingular, $Ax = b$ always has a unique solution given explicitly as $x = A^{-1}b$.

In the general case ($m \neq n$) the homogeneous system may still have exactly one solution (the trivial one) or infinitely many, and the non-homogeneous system may still have no solution, exactly one solution, or infinitely many. However, in this case a unique solution cannot be expressed in the form $x = A^{-1}b$, since A^{-1} is not defined.

The algorithm to solve a general linear system of m equations in n unknowns consists of applying Gaussian elimination to the <u>augmented</u> matrix $[\, A \mid b \,]$ until it reaches a so called <u>row echelon form</u>:

$$\begin{pmatrix} \dagger & \dots & * & \dots & \dots & \dots & \dots & * \\ 0 & \dots & 0 & \dagger & \dots & \dots & \dots & * \\ 0 & \dots & 0 & \dots & 0 & \dagger & \dots & * \\ 0 & \dots & 0 & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & \dots & \dots & \dots & 0 \end{pmatrix}$$

where the coefficients labeled $\dagger$ represent the pivots and those labeled $*$ represent arbitrary real numbers.

The row echelon form is characterized by the following two conditions:

(i) the first non-zero coefficient in each row (pivot) must appear in a column strictly to the right of the pivot in the previous row;

(ii) every coefficient in a row below and in the column of or to the left of a pivot must be zero.

Definition: The free variables are the unknowns corresponding to the columns where no pivots are located. The pivot variables are the unknowns corresponding to the columns where the pivots are located.

The row echelon system is solved by back substitution (from bottom up) assigning parameter values (s, t, etc.) to the free variables and solving for the pivot variables in term of the free.

In case the system is non-homogeneous, it is inconsistent if, and only if the row echelon form of the augmented matrix $[\,A \mid b\,]$ has in its last nonzero row only one nonzero coefficient, located on the last $(n+1)-$st column.

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

First we find the row echelon form of the augmented matrix $[\,A \mid b\,]$.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 4 & 5 & 6 & 3 \\ 3 & 6 & 7 & 8 & 5 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & -2 & -4 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The pivots are 1 and -1, corresponding to the pivot variables x_1 and x_3. The free variables are x_2 and x_4, since the second and fourth columns of the row echelon form do not contain pivots. We solve by back substitution:

$$x_4 = t, \quad -x_3 = 1 + 2x_4 \text{ gives } x_3 = -2t - 1$$
$$x_2 = s, \quad x_1 = 1 - 2x_2 - 3x_3 - 4x_4 = 4 - 2s - 2t$$

Now we write the general solution:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ -2 \\ 1 \end{pmatrix}}_{x_{\text{homogeneous}}} + \underbrace{\begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix}}_{x_{\text{particular}}}$$

Definition: The number of nonzero rows in the row echelon form of a matrix A (which equals the number of pivots) is called the (row) rank of the matrix.

Remark: A square row echelon matrix is always upper triangular.

We can extend the concept of upper triangular to $m \times n$ matrices $A = (a_{ij})$ that are not square, by still requiring $a_{ij} = 0$ for $j < i$.

Just as we did for square matrices, we may realize the elementary row operations that lead from A to its row echelon form U through multiplication on the left by elementary matrices. If no row exchanges are needed, this leads to the factorization

$$A = LU,$$

where $L \in \mathcal{M}_{m\times m}$ is lower triangular with 1's on the diagonal, and $U \in \mathcal{M}_{m\times n}$ is a row echelon matrix.

If row exchanges are needed, they are done before the factorization by multiplying A on the left by a permutation matrix $P \in \mathcal{M}_{m\times m}$. This leads to

$$PA = LU.$$