MA 511, Session 8

Linear Independence and Spanning

Let V be a real vector space, and let $v_1, \dots v_k \in V$.

Definition: The vectors $v_1, \dots v_k$ are linearly independent if the only linear combination of them that gives the 0 vector in V is the trivial one, i.e.

$$c_1 v_1 + \cdots + c_k v_k = 0 \Rightarrow c_1 = \cdots = c_k = 0.$$

If there is a nontrivial linear combination of them that gives the 0 vector, they are called linearly dependent.

Definition: We say $v_1, \dots v_k$ span V if every vector in V is a linear combination of $v_1, \dots v_k$.

Definition: $v_1, \dots v_k$ are a basis of V if they are linearly independent and they span V.

Remark: $v_1, \ldots v_k$ are linearly dependent if, and only if one of them is a linear combination of the others.

Necessity: Assume $v_1, \ldots v_k$ linearly dependent. Then $c_1 v_1 + \cdots + c_k v_k = 0$ where at least one of the coefficients c_i is not equal to zero. Without loss of generality, we may assume $c_1 \neq 0$ (otherwise we reorder the vectors $v_1, \ldots v_k$). We may now solve for v_1 and see that $v_1 = -\frac{c_2}{c_1} v_2 - \cdots - \frac{c_k}{c_1} v_k$ is a linear combination of $v_2, \ldots, v_k$.

Sufficiency: Assume, without loss of generality, that v_1 is a linear combination of $v_2, \ldots v_k$ (otherwise, reorder the vectors $v_1, \ldots v_k$). Then we may write $v_1 = c_2 v_2 + \cdots + c_k v_k$. It follows that

$$1 v_1 - c_2 v_2 - \cdots - c_k v_k = 0,$$

and the first coefficient in the linear combination is not zero. Thus $v_1, \ldots, v_k$ are linearly dependent.

Example: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ are linearly dependent since

$$1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 1 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and at least one of the coefficients 1,-2,1 is not zero (in fact, all).

Example: If one of the vectors $v_1, \ldots v_k$ is the zero vector, then $v_1, \ldots v_k$ are always linearly dependent. Indeed, we may assume without loss of generality that $v_1 = 0$ (otherwise, reorder the vectors $v_1, \ldots v_k$) and then $1v_1 + 0v_2 + \cdots + 0v_k = 0$, where the first coefficient in the linear combination is not zero.

Remark: Suppose a $m \times n$ matrix A is taken to row echelon form U by elementary row operations. Then the nonzero rows of U and the columns containing pivots are linearly independent.

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now $\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 0 & -4 & -8 & -12 \end{pmatrix}$ are linearly independent since

$$r \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} + s \begin{pmatrix} 0 & -4 & -8 & -12 \end{pmatrix} = 0$$

implies $1r + 0s = 0$, hence $r = 0$ and $s \begin{pmatrix} 0 & -4 & -8 & -12 \end{pmatrix} = 0$. Thus, $s = 0$. This means that, if a linear combination of the first two rows of U gives the zero vector then it is necessarily the trivial combination, which—by definition—means those two rows are linearly independent.

You can now convince yourself that this same argument applies to any triangular matrix (upper or lower) and, more generally, to any row echelon matrix.

Similarly,

$$r\begin{pmatrix}1\\0\\0\end{pmatrix}+s\begin{pmatrix}2\\-4\\0\end{pmatrix}=0$$

implies $0r-4s=0$, hence $s=0$, then $1r+2s=0$ reads $r=0$. This means that the two column vectors of U containing the pivots,

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}2\\-4\\0\end{pmatrix},$$

are linearly independent.

Again, you may now convince yourself that the same argument applies to the columns containing pivots of any row echelon matrix.

Remark: The columns of a $m \times n$ matrix A are linearly independent in $\mathbb{R}^n$ if, and only if the only solution to $Ax = 0$ is the trivial one, $0 \in \mathbb{R}^m$:

$$Ax = \begin{pmatrix} \vec{C}_1 & \vec{C}_2 & \dots & \vec{C}_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$
$$= x_1\vec{C}_1 + x_2\vec{C}_2 + \cdots + x_n\vec{C}_n = 0$$

means that nontrivial solutions to $Ax = 0$ exist if, and only if nontrivial linear combinations of the columns of A that give the zero vector exist.

Remark: The rows of A are independent if, and only if the only solution to $A^T x = 0$ is the trivial solution $0 \in \mathbb{R}^m$.

Theorem: If $v_1, \ldots, v_n \in \mathbb{R}^m$ and $n > m$, then they are linearly dependent.

<u>Proof</u>: let $v_1, \ldots, v_n$ be the columns of a matrix A and consider the system $Ax = 0$. Now A has more columns than rows ($n > m$) and thus its row echelon form has at most m pivots. Hence, when solving $Ax = 0$ there are at least $n - m$ free variables giving infinitely many nontrivial solutions to this system. As we just saw, any such solution leads to a nontrivial linear combination of $v_1, \ldots, v_n$ that gives the zero vector, showing the linear dependence of $v_1, \ldots, v_n$.

Example: The simplest vectors that span $\mathbb{R}^3$ are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Indeed, $xe_1 + ye_2 + ze_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ for any $x, y, z \in \mathbb{R}$.

Of course, a set like $e_1, e_2, e_3, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -4 \\ 3 \end{pmatrix}$, also span $\mathbb{R}^3$, but the last two vectors are superfluous.

Definition: Given $v_1, \ldots, v_k \in V$ the set W of all their linear combinations is the <u>linear span</u> of $v_1, \ldots, v_k$.

Lemma: The linear span W of $v_1, \ldots, v_k$ is a subspace of V

<u>Proof</u>: (i) $0 = 0v_1 + \cdots + 0v_k \in W$;

(ii) Let $w_1, w_2 \in W$. Then, $w_1 = c_1 v_1 + \cdots + c_k v_k$ and $w_2 = d_1 v_1 + \cdots + d_k v_k$. Hence,

$$w_1 + w_2 = (c_1 + d_1)v_1 + \cdots + (c_k + d_k)v_k \in W.$$

(iii) Let $w \in W$ and $r \in \mathbb{R}$.

Then, $w = c_1 v_1 + \cdots + c_k v_k$ and thus

$$rw = (rc_1)v_1 + \cdots + (rc_k)v_k \in W.$$

Example: Is the vector $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ in the space generated by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$?

Solution: We need to solve the system

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Use Gaussian elimination on the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 1 \\ 3 & 6 & 9 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 1 \\ 0 & -6 & -12 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We see that the system is consistent and has infinitely many nontrivial solutions

$$x_3 = t, \quad x_2 = -2t - \tfrac{1}{3}, \quad x_1 = t + \tfrac{4}{3}.$$

Summarizing:

If $v_1, \ldots, v_k$ are vectors in $\mathbb{R}^n$, we can check to see if they are linearly independent by solving $Ax = \begin{pmatrix} v_1 & \ldots & v_k \end{pmatrix} x = 0$, where $v_1, \ldots, v_k$ are used as columns of a matrix A. If the only solution is the trivial solution $x = 0 \in \mathbb{R}^k$ they are linearly independent. If there are nontrivial solutions (i.e. the row echelon form of A has fewer than k pivots) then they are linearly dependent.

To see if a vector $b \in \mathbb{R}^n$ lies in the column space of a matrix A, check $Ax = b$. If it has a solution, then b is in the column space of A.