# MA 511, Session 10

## <u>The Four Fundamental Subspaces of a Matrix</u>

Let $A$ be a $m \times n$ matrix.

(i) The <u>row space</u> $\mathcal{C}(A^T)$ of $A$ is the subspace of $\mathbb{R}^n$ spanned by the rows of $A$.

(ii) The <u>null space</u> $\mathcal{N}(A)$ of $A$ is the subspace of $\mathbb{R}^n$ of solutions of $Ax = 0$.

(iii) The <u>column space</u> $\mathcal{C}(A)$ of $A$ is the subspace of $\mathbb{R}^m$ spanned by the columns of $A$.

(iv) The <u>left null space</u> $\mathcal{N}(A^T)$ of $A$ is the subspace of $\mathbb{R}^m$ of solutions of $A^T x = 0$.

Let us discuss how to find bases for these vector spaces and determine their dimensions.

(i) We consider the row space first. Note that if $B$ is obtained from $A$ by elementary row operations, then the rows of $B$ are linear combinations of the rows of $A$. Since the elementary row operations can all be reversed, we also have the rows of $A$ are linear combinations of the rows of $B$, i.e. $\mathcal{C}(A^T) = \mathcal{C}(B^T)$. In particular, if we take $A$ to its row echelon form $U$ by elementary row operations, then the nonzero rows of $U$ form a basis for its row space and hence for the row space of $A$. If we let $r = \text{rank } A$, then

$$\dim \mathcal{C}(A^T) = r.$$

(ii) As for the null space, we can quickly see that elementary row operations do not change this space: if $A = LU$ with $L$ lower triangular with all coefficients on its diagonal equal to 1, then

$$Ax = 0 \iff Ux = 0.$$

Indeed,

$$Ax = 0 \iff LUx = 0 \iff L^{-1}LUx = L^{-1}0 = 0.$$

Then, a basis for $\mathcal{N}(A)$ is actually found as a basis for $\mathcal{N}(U)$ using the general method we described in session 9 from the free variables in the system $Ux = 0$. If rank $A = r$, then the number of pivots is $r$ and the number of free variables is $n - r$. Thus,

$$\dim \mathcal{N}(A) = n - r.$$

(iii) We now turn to the column space. Since elementary row operations do change the column space, it is not obvious even that $\dim \mathcal{C}(A) = \dim \mathcal{C}(A^T)$. This follows from the following nontrivial observation:

Suppose that $U$ is the row echelon form of $A$. Then $U = EA$, where $E$ is a product of elementary matrices. Let us write this relation in terms of the columns of $U$, $u_1, \dots, u_n \in \mathbb{R}^n$ and those of $A$, $v_1, \dots, v_n \in \mathbb{R}^n$

$$\begin{aligned}\begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix} &= E \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix} \\ &= \begin{pmatrix} Ev_1 & Ev_2 & \dots & Ev_n \end{pmatrix}\end{aligned}$$

that is, $Ev_j = u_j$ for $1 \le j \le n$. We shall prove below that invertible matrices transform linearly independent vectors into linearly independent vectors.

Hence, as we know that the columns of $U$ that contain the pivots are linearly independent, it follows that <u>the corresponding columns of $A$</u> are also linearly independent—and the other columns of $A$ are dependent from these—so that they also span $\mathcal{C}(A)$ and thus they form a basis for this space. Hence,

$$\dim \mathcal{C}(A) = r.$$

**Lemma:** Assume $E$ is a nonsingular matrix. Then,

$$v_1, \ldots, v_k \ \text{ l.i.} \quad \Longleftrightarrow \ Ev_1, \ldots, Ev_k \ \text{ l.i.}$$

<u>Proof</u>: **Necessity (only if):** Let $c_1 E v_1 + \cdots + c_k E v_k = 0$. We must prove that $c_1 = \cdots = c_k = 0$. First note that $E(c_1 v_1 + \cdots + c_k v_k) = 0$. Set $x = c_1 v_1 + \cdots + c_k v_k$ and we immediately see that $Ex = 0$. Since $E$ is nonsingular, this implies $0 = E^{-1}0 = E^{-1}Ex = x$, that is $x = c_1 v_1 + \cdots + c_k v_k = 0$. Since $v_1, \ldots, v_k$ are linearly independent, then $c_1 = \cdots = c_k = 0$, proving the linear independence of $Ev_1, \ldots, Ev_k$.

**Sufficiency (if):** Assume $u_1 = Ev_1, \ldots, u_k = Ev_k$ linearly independent and let $v_1 = E^{-1}u_1, \ldots, v_k = E^{-1}u_k$. We need to prove $v_1, \ldots, v_k$ are linearly independent. We can reverse the roles of $v_1, \ldots, v_k$ and $Ev_1, \ldots, Ev_k$ in the necessity part and see (using $E^{-1}$ in place of $E$) that $u_1, \ldots, u_k$ linearly independent implies $E^{-1}u_1, \ldots, E^{-1}u_k$ linearly independent, which completes the proof.

(iv) As for the left null space $\mathcal{N}(A^T)$, it is clear that

$$\dim \mathcal{N}(A^T) = m - r,$$

since rank $A =$ rank $A^T$. To find a basis for $\mathcal{N}(A^T)$, we take $A^T$ to its row echelon form and using the $m - r$ free variables generate $m - r$ basis vectors in the usual way (giving each free variable in turn the value 1 while all other independent variables take on the value 0).

**Example:** Find bases and the dimensions of the four fundamental spaces of

$$A = \begin{pmatrix} 0 & 2 & 3 & 4 \\ 0 & 6 & 7 & 8 \\ 0 & 10 & 11 & 12 \end{pmatrix}.$$

Solution: We find the row echelon form of $A$:

$$\begin{pmatrix} 0 & 2 & 3 & 4 \\ 0 & 6 & 7 & 8 \\ 0 & 10 & 11 & 12 \end{pmatrix} \to \begin{pmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & -4 & -8 \end{pmatrix} \to \begin{pmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) Basis for row space $\mathcal{R}(A^T)$: $\begin{pmatrix} 0 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ -4 \end{pmatrix}.$

(iii) Basis for column space $\mathcal{R}(A)$: $\begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix}.$

(ii) Basis for $\mathcal{N}(A)$: We note that in solving $Ux = 0$ the free variables are $x_1$ and $x_4$. Using $x_1 = 0$ and $x_4 = 1$ we obtain $x_3 = -2$ and $x_2 = \frac{-3(-2)-4(1)}{2} = 1$, giving the vector $\begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}$. Next we take $x_1 = 1$ and $x_4 = 0$, which give the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Thus,

basis for $\mathcal{N}(A)$: $\begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

We have just established that

$$\dim \mathcal{R}(A^T) = \dim \mathcal{R}(A) = \dim \mathcal{N}(A) = 2.$$

(iv) Basis for $\mathcal{N}(A^T)$: Since $\dim \mathcal{N}(A^T) = 3-2 = 1$, it follows that in order to find a basis for the left null space of $A$ we only need one nonzero vector $x \in \mathbb{R}^3$ such that $A^T x = 0$. We take $A^T$ to row echelon form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{pmatrix} \to \begin{pmatrix} 2 & 6 & 10 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 6 & 10 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and now see that $x_3 = 1$ gives $x_2 = -2$ and $x_1 = 1$.

Thus, basis for $\mathcal{N}(A^T)$: $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.