MA 511, Session 11

Graphs and Kirchhoff's Laws

We'll call a graph a set of points (called the <u>nodes</u>) and oriented segments joining them (called <u>edges</u>). A <u>path</u> is any sequence of consecutive edges. We 'll only consider graphs that are <u>connected</u>, i.e. such that there is a path (not necessarily unique) from any node to any other.

The Incidence Matrix: Consider a graph with n nodes and m edges. We define the $m \times n$ edge-node incidence matrix A row-by-row as follows: If the *i*-th edge originates at node j and ends at node k, then the *i*-th row of $A = (a_{il})$ contains zeros in all its components, except for $a_{ij} = -1$ and $a_{ik} = 1$.

Example: Consider a graph consisting of 4 nodes and the following 5 edges, e_{12} , e_{23} , e_{34} , e_{14} , e_{13} (where e_{ij} means the edge that originates at the *i*-th node and ends at the *j*-th node). Then, the edgenode incidence matrix is

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

We now turn to the calculation and interpretation of the four fundamental subspaces for the incidence matrix.

1) Null Space: We immediately see that the column vector with all its components equal to 1 is in the null space since any row of A multiplied by it results in (-1)1 + (1)1 = 0. It turns out that any vector in the null space must be a multiple of this one and

thus dim
$$\mathcal{N}(A) = 1$$
 and a basis for $\mathcal{N}(A)$ is $\begin{pmatrix} 1\\1\\1\\1 \end{pmatrix}$.

2) Column space: First recall that the column space consists of all linear combinations of the columns of A, i.e. all products Ac where $c \in \mathbb{R}^n$. If we interpret the graph as a circuit and the components of c as potentials (e.g. electric potential) located at the corresponding nodes, then the structure of the rows of Amakes the product Ac have for components the *differences of potential* across the edges, so that Ax = bcan be solved if, and only if these differences in potential across edges satisfy the necessary constraints to make b belong to the column space of A.

By direct observation of A we see that the sum of the first and second rows gives the fifth row, and the sum of the third and fifth rows gives the fourth row. Thus, in order for b to be in the column space, it must satisfy $b_1 + b_2 = b_5$ and $b_3 + b_5 = b_4$, or equivalently, $b_1 + b_2 - b_5 = 0$ and $b_3 - b_4 + b_5 = 0$. A basis for the solution space of this system is

$$v_{1} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_{2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, v_{3} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},$$

and $\dim \mathcal{C}(A) = 3$.

It is worth noting that v_1 is the opposite of the second column of A, v_2 is the fourth column of A, and v_3 is the sum of the second and third columns of A.

3) Left-Null Space:

Definition: A <u>loop</u> is a path that begins and ends at the same node (irrespective of the orientation of the edges involved).

We can write down any loop by just indicating in order the nodes it goes through.

In the example we are examining, 1-2-3-1 is a loop (since we have edges connecting nodes 1 and 2,

nodes 2 and 3, and nodes 3 and 1). Another loop is 1-3-4-1, and yet another one is 1-2-3-4-1. However, this last one is just a superposition of the first two loops mentioned. Let us now assign a column vector to each loop as follows: if edge i is traversed in the loop in the direction of its orientation, then the i-th component of the column vector is 1; if edge i is traversed in the loop in the loop in the direction opposite of its orientation, then the i-th component of the column vector is -1; if edge i is not in the loop, then the i-th component of the column vector is 0.



Note that these vectors, y_1 and y_2 , have for compo-

nents precisely the coefficients in the two equations for b we just solved to find the column space and thus $y^T A = 0$ for $y = y_1$ and $y = y_2$. Therefore, y_1, y_2 form a basis for $\mathcal{N}(A^T)$ and dim $\mathcal{N}(A^T) = 2$.

This is no coincidence. It is, in fact,

Kirchhoff's Voltage Law: The sum of potential differences around a loop must be zero.

4) Row Space: Rows in the edge-node incidence matrix are independent if, and only if the edges they correspond to contain no loops. So, it is very easy to see from the graph that, e.g. the first three rows are linearly independent since edges 1, 2, and 3 give the path 1-2-3-4, which contains no loops.

Of course, in this example, the 3×4 matrix consisting of the first three rows of A is in row echelon form and thus its rows are linearly independent (for "free"). At any rate,

 $x_1 = (-1, 1, 0, 0), \ x_2 = (0, -1, 1, 0), \ x_3 = (0, 0, -1, 1)$ form a basis for $\mathcal{C}(A^T)$ and dim $\mathcal{C}(A^T) = 3$. We finally note that a vector f is in the row space of A (or, equivalently, in the column space of A^T) if, and only if $f^T x = 0$ for all $x \in \mathcal{N}(A)$, i.e. if, and only if $f_1 + f_2 + f_3 + f_4 = 0$. The components of f can be interpreted as the "current sources" at the nodes.

It turns out that A^T is the right matrix for currents, and all that this condition says is

Kirchhoff's Current Law: The net current into every node is zero.

Let us stress once more that, once we have an edge-node incidence matrix, we can *algebraically* find bases for and the dimensions of the four fundamental subspaces using the general methods we gave in session 10. However, using the graph corresponding to the matrix, many of those bases can be *visually* deduced from Kirchhoff's laws.