MA 511, Session 12

Linear Transformations

Let V and W be real vector spaces.

Definition: A function $f : V \longrightarrow W$ is a linear transformation if, for any vectors $v_1, v_2 \in V$ and any scalar $c \in \mathbb{R}$,

$$f(v_1 + v_2) = f(v_1) + f(v_2),$$
$$f(cv_1) = cf(v_1).$$

Equivalently, $f(c_1v_1 + c_2v_2) = c_1f(v_1) + c_2f(v_2)$ for any scalars c_1, c_2 and any vectors v_1, v_2.

Example: Let $f : \mathcal{P}_n \longrightarrow \mathcal{P}_{n-2}$ be the *second derivative* function,

$$f(p) = p''.$$

Then, f is a linear transformation of $V = \mathcal{P}_n$ into $W = \mathcal{P}_{n-2}$.

Proof: Let $v_1, v_2 \in \mathcal{P}_n$ and $c_1, c_2 \in \mathbb{R}$. Then,

$$f(c_1v_1 + c_2v_2) = (c_1v_1 + c_2v_2)''$$
$$= c_1v_1'' + c_2v_2'' = c_1f(v_1) + c_2f(v_2).$$

For example, if $p(t) = 2t^3 - t^2 + 4$ and $q = f(p)$, then $q(t) = 12t - 2$.

Example: Let $L : \mathcal{C}[0,1] \longrightarrow \mathbb{R}$ be the definite integral

$$L(f) = \int_0^1 f(t)\,dt.$$

Then, L is a linear transformation of $V = \mathcal{C}[0,1]$ into $W = \mathbb{R}$.

Proof: Let $v_1, v_2 \in \mathcal{C}[0,1]$ and $c_1, c_2 \in \mathbb{R}$. Then,

$$\begin{aligned} L(c_1v_1 + c_2v_2) &= \int_0^1 (c_1v_1 + c_2v_2)(t)\,dt \\ &= c_1 \int_0^1 v_1(t)\,dt + c_2 \int_0^1 v_2(t)\,dt \\ &= c_1L(v_1) + c_2L(v_2). \end{aligned}$$

For example, for $f(x) = x^2$, $L(f) = \frac{1}{3}$.

Example: Let $T : \mathcal{C}[0,1] \longrightarrow \mathbb{R}$ be the evaluation function at $\frac{1}{4}$, $T(f) = f(\frac{1}{4})$. Then, T is a linear transformation of $V = \mathcal{C}[0,1]$ into $W = \mathbb{R}$.

Proof: Let $v_1, v_2 \in \mathcal{C}[0,1]$ and $c_1, c_2 \in \mathbb{R}$. Then,

$$\begin{aligned} T(c_1v_1 + c_2v_2) &= (c_1v_1 + c_2v_2)(\tfrac{1}{4}) \\ &= c_1v_1(\tfrac{1}{4}) + c_2v_2(\tfrac{1}{4}) \\ &= c_1T(v_1) + c_2T(v_2). \end{aligned}$$

Example: Let A be a $m \times n$ matrix, and let $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the transformation $L(x) = Ax$. Then, L is a linear transformation of $V = \mathbb{R}^n$ into $W = \mathbb{R}^m$.

Proof: Let $v_1, v_2 \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$. Then,

$$\begin{aligned} L(c_1v_1 + c_2v_2) &= A(c_1v_1 + c_2v_2) \\ &= c_1Av_1 + c_2Av_2 \\ &= c_1L(v_1) + c_2L(v_2). \end{aligned}$$

Theorem: Let $A = (a_{ij})$ and $B = (b_{jk})$ be $m \times n$ and $n \times p$ matrices, and let $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $M : \mathbb{R}^p \longrightarrow \mathbb{R}^n$ be the corresponding transformations $L(x) = Ax$ and $M(y) = By$. Then the composition $L \circ M : \mathbb{R}^p \longrightarrow \mathbb{R}^m$ of L and M, defined as $(L \circ M)(y) = L(M(y))$, corresponds to the product AB of the matrices A and B.

<u>Proof</u>: For $y = (y_1, \ldots, y_p)$, the components of $x = M(y) = By$ are $x_j = \sum_{k=1}^{p} b_{jk} y_k$. The components of $b = L(M(y)) = L(x) = Ax$ are

$$b_i = \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ij} b_{jk} y_k = \sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right) y_k .$$

Hence $b = ABy$.

A linear transformation is entirely determined by its action on a basis.

Suppose that $B = \{v_1, \ldots, v_n\}$ is a basis for V and $B' = \{w_1, \ldots, w_m\}$ is a basis for W. Let $L : V \longrightarrow W$ be a linear transformation. It follows that, for $x \in V$,

$$L(x) = L(c_1 v_1 + \cdots + c_n v_n) = c_1 L(v_1) + \cdots + c_n L(v_n).$$

We can represent a linear transformation "via bookkeeping" by a matrix. Let

$$L(v_j) = a_{1j} w_1 + \cdots + a_{mj} w_m, \quad 1 \leq j \leq n.$$

We now say that the $m \times n$ matrix $A = (a_{ij})$ is the <u>matrix of linear transformation L in the bases B,B'</u>. We use the following notation:

$$A = (L)_{BB'}$$

Then, A transforms the components of a vector v in V with respect to the basis B into the components of $L(v)$ in W with respect to the basis B'.

Since

$$v_j = \sum_{i=1}^{n} \delta_{ij} v_i,$$

where δ_{ij} is Kronecker's symbol, we see that the components of the vector v_j in the basis B are given by the j-th standard basis vector e_j of $\mathbb{R}^n$. Hence,

$$Ae_j = C_j,$$

the j-th column of A gives, by definition of the matrix, the components of $L(v_j)$ with respect to the basis B'. For an arbitrary vector $v \in V$, the claim follows by taking linear combinations of $v_1, \ldots, v_n$.

Example: Let $V = \mathcal{P}_4$ and $W = \mathcal{P}_3$. Consider the linear transformation $T : V \longrightarrow W$ given by $T(p) = p' + p''$. Let $B = \{1, t, t^2, t^3, t^4\}$ and $B' = \{1, t, t^2, t^3\}$ be bases for V and W, respectively. Then, the matrix representing T in these bases is

$$A = (T)_{BB'} = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

If $p(t) = -2t^4 + 3t^2 - 6t + 7$, then $q = T(p)$ is the cubic polynomial

$$q(t) = (-8t^3 + 6t - 6) + (-24t^2 + 6) = -8t^3 - 24t^2 + 6t.$$

To compute this using the matrix of the linear transformation, we have to multiply it by the vector of components of p in the basis B, i.e. $(7, -6, 3, 0, -2)^T$. We obtain

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ -6 \\ 3 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ -24 \\ -8 \end{pmatrix}.$$

This product gives the components or q in the basis B', so that $q(t) = (0)1 + (6)t + (-24)t^2 + (-8)t^3 = -8t^3 - 24t^2 + 6t$, just as the analytical result.

Example: Let V, W, and L be as in the last example. Find the null space of $A = (L)_{BB'}$ and interpret it in terms of the vector space V.

<u>Solution</u>: The null space of A consists of the vectors $x \in \mathbb{R}^n$ such that $Ax = 0$. Now, $Ax = 0$ if, and only if $L(v) = 0$, where $v = x_1 v_1 + \cdots + x_n v_n$.

We can find the general solution of the linear, homogeneous, ordinary differential equation with constant coefficients

$$L(u) = u'' + u' = 0$$

by writing the characteristic polynomial $r^2 + r = r(r+1)$, and using its roots 0 and -1. We see that the solution set S is a 2-dimensional subspace of $\mathcal{C}^2(-\infty, \infty)$ (a plane in that vector space),

$$u(t) = c_1 e^{0t} + c_2 e^{-1t} = c_1 + c_2 e^{-t},$$

and the only functions in S that are also in V are the constants $u(t) = c_1$. Thus, the null space of L consists of the constants (polynomials of degree 0). Computing algebraically the solution set of $Ax = 0$, we find that it is a line in $\mathbb{R}^5$, given parametrically as

$$l = \left\{ s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ s \in \mathbb{R} \right\}.$$

This corresponds to the constant polynomials.

Example: Plane rotations about the origin by any angle α are linear transformations in $V = W = \mathbb{R}^2$, represented by orthogonal matrices

$$(Q_\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Example: Projections onto a line are linear transformations in $V = W = \mathbb{R}^2$, with

$$(P_0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (P_{\pi/2}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

representing, respectively, the projections onto the x-axis and the y-axis.

Example: Reflections across a line are linear transformations in $V = W = \mathbb{R}^2$, with

$$(R_{\pi/4}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

representing the reflection across the line $x = y$.