MA 511, Session 13

Review

Example 1: Let $A \in \mathcal{M}_{n \times n}$. Then A is nonsingular if, and only if

i) A^{-1} exists

ii) $\text{rank}(A) = n$

iii) A^T is nonsingular

iv) $Ax = 0$ has only the trivial solution $x = 0$

v) $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$

vi) the rows of A are linearly independent

vii) the columns of A are linearly independent

viii) $\mathcal{C}(A) = \mathbb{R}^n$

ix) $\mathcal{C}(A^T) = \mathbb{R}^n$

x) the row echelon form of A has nonzero diagonal entries

xi) A is the product of elementary matrices

Example 2: Find a basis for the space W spanned by

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} \quad \begin{pmatrix} 9 \\ 10 \\ 11 \\ 12 \end{pmatrix}$$

We put these vectors as rows of a matrix A and take it to row echelon form:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that a basis for the space W spanned by those four vectors is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

and therefore, $\dim W = 2$.

Alternatively, put these vectors as columns of a matrix A^T:

$$A^T = \begin{pmatrix} 1 & 1 & 5 & 9 \\ 1 & 2 & 6 & 10 \\ 1 & 3 & 7 & 11 \\ 1 & 4 & 8 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 5 & 9 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Columns 1 and 2 are the pivot columns. The corresponding columns of A^T form a basis of W:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Example 3: Find bases and the dimensions of $\mathcal{C}(A)$ and $\mathcal{N}(A)$ in the previous example.

<u>Proof</u>: A basis for $\mathcal{C}(A)$ is formed by the columns that correspond to those of its row echelon form which contain pivots:

$$\begin{pmatrix} 1 \\ 1 \\ 5 \\ 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 6 \\ 10 \end{pmatrix}$$

Alternatively, one can take nonzero rows of the row echelon form of A^T:

$$\begin{pmatrix} 1 \\ 1 \\ 5 \\ 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

A basis for $\mathcal{N}(A)$ is found by solving the system $x_1+x_2+x_3+x_4 = 0$ and $x_2+2x_3+3x_4 = 0$. We have x_3 and x_4 as free variables so that $\dim \mathcal{N}(A) = 2$, and a basis is, e.g.

$$\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

Example 4: Let $V = \mathcal{C}[0,1]$. Which of the following functions $T : V \longrightarrow \mathbb{R}$ are linear transformations?

(i) $$T(f) = \int_0^1 f(x)e^x \, dx$$

(ii) $$T(f) = \int_0^1 f(x)^2 \, dx$$

(iii) $$T(f) = f(\tfrac{1}{2}) + \int_0^1 f(x) \, dx$$

(iv) $$T(f) = 1 + \int_0^1 f(x) \, dx$$

(i) and (iii) are linear transformations (since you can check that $T(c_1 f + c_2 g) = c_1 T(f) + c_2 T(g)$ for $c_1, c_2 \in \mathbb{R}$ and $f, g \in V$), (ii) and (iv) are not (e.g., $T(0) \neq 0$ in (iv)).

Example 5: From session 12 we use $V = \mathcal{P}_4$, $W = \mathcal{P}_3$, and $T : V \longrightarrow W$ defined by $T(p) = p'' + p'$.

Using the basis $B = \{1, t, t^2, t^3, t^4\}$ for V and $B' = \{1, t, t^2, t^3\}$ for W, the matrix A for the linear transformation T using these bases is

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

To find, for example, $T(3t^3 - 2t^2 + 5)$, we multiply the coefficient vector of $p(t) = 3t^3 - 2t^2 + 5$ in the basis B, $x = \begin{pmatrix} 5 & 0 & -2 & 3 & 0 \end{pmatrix}^T$ by A on the left, and we obtain the coefficients of the $T(p)$ in the basis B':

$$Ax = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 6 & 0 \\ 0 & 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ -2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 14 \\ 9 \\ 0 \end{pmatrix}.$$

Thus, $T(p) = -4(1) + 14(t) + 9(t^2)$, as is readily verified by computing $p'' + p'$.

Regarding $\mathcal{N}(A)$, we find immediately that $\dim \mathcal{N}(A) = 1$, since A is in row echelon form. Moreover, a basis for $\mathcal{N}(A)$ consists of the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, representing the polynomial 1, which coincides with the fact that the only polynomials for which the sum of the first and second derivative vanishes are those of degree zero (constants):

We solve the differential equation $u'' + u' = 0$ from its characteristic polynomial

$$r^2 + r = r(r+1),$$

with characteristic roots 0 and -1. The general solution is

$$u(x) = c_1 + c_2 e^{-x},$$

and the only polynomials among these functions are the constants.

Example 6: Write an LU decomposition for the matrix A in Example 2.

The elimination process in Example 2 was

$$E_{42}(-1)E_{32}(-1)E_{41}(-9)E_{31}(-5)E_{21}(-1)A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$A = E_{21}(1)E_{31}(5)E_{41}(9)E_{32}(1)E_{42}(1) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ 9 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 7: Let V be the set of all 2×2 singular matrices. Is V a vector pace?

No, because $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in V$, but

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I} \notin V.$$

Example 8: Let V be a vector space and $v_1, v_2, v_3 \in V$. Define

$$w_1 = v_1 - v_2, \quad w_2 = v_2 - v_3, \quad w_3 = v_3 - v_1.$$

Prove that w_1, w_2, w_3 are linearly dependent.

<u>Proof</u>: Note that

$$1w_1 + 1w_2 + 1w_3 = (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_1) = 0,$$

and the coefficients in the linear combination are not all zero (in fact, none of them are). Therefore, this is a dependence relation.

Example 9: Let $V = \mathcal{C}^2(-\infty, \infty)$ be the vector space of functions having two continuous derivatives at all $t \in \mathbb{R}$, and let $W = \mathcal{C}(-\infty, \infty)$. Define the linear transformation $L : V \longrightarrow W$ as $Lf = f'' - f$. Find a basis for $\mathcal{N}(L)$.

<u>Solution</u>: Note that $\mathcal{N}(L)$ is the set of all solutions of the differential equation $f'' - f = 0$.

We know that the characteristic polynomial is $r^2 - 1 = (r + 1)(r - 1)$, with roots -1 and 1. Thus, the general solution is

$$f(t) = c_1 e^{-t} + c_2 e^{t}.$$

We see that $\dim \mathcal{N}(L) = 2$ and a basis for it consists of the two vectors

$$f_1(t) = e^{-t},$$
$$f_2(t) = e^{t}.$$

Example 10: Let $W = \{p \in \mathcal{P}_3 : \int_0^1 p(t)\,dt = 0\}$. Prove that W is a subspace of $V = \mathcal{P}_3$, and find a basis for it.

<u>Solution</u>: (i) $0 \in W$ since $\int_0^1 0\,dt = 0$.

(ii) Assume $f, g \in W$. Then,
$\int_0^1 (f+g)(t)\,dt = \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt = 0 + 0 = 0.$

(iii) Assume $f \in W$ and $c \in \mathbb{R}$. Then,
$$\int_0^1 (cf)(t)\,dt = c\int_0^1 f(t)\,dt = c\,0 = 0.$$

To find a basis, let us write the general vector in V as

$$p(t) = a_3t^3 + a_2t^2 + a_1t + a_0,$$

and determine the necessary condition(s) for it to be in W. We have

$$\begin{aligned}\int_0^1 p(t)\,dt &= a_3\frac{t^4}{4} + a_2\frac{t^3}{3} + a_1\frac{t^2}{2} + a_0t\Big|_0^1 \\ &= \frac{1}{4}a_3 + \frac{1}{3}a_2 + \frac{1}{2}a_1 + a_0 = 0\end{aligned}$$

This is one equation in three unknowns, and so we see that $\dim W = 3$ and a basis is found by taking a_1, a_2, a_3 as parameters and giving them in turns the value 1 to each while the others take on the value 0. Thus,

$$(\,-\tfrac{1}{2} \quad 1 \quad 0 \quad 0\,);(\,-\tfrac{1}{3} \quad 0 \quad 1 \quad 0\,);(\,-\tfrac{1}{4} \quad 0 \quad 0 \quad 1\,)$$

give the coefficients of a basis for W in the selected basis for V. Thus,

$$\begin{aligned} p_1(t) &= t - \tfrac{1}{2} \\ p_2(t) &= t^2 - \tfrac{1}{3} \\ p_3(t) &= t^3 - \tfrac{1}{4} \end{aligned}$$

form a basis for W.