MA 511, Session 18

Orthogonal Bases and Function Spaces

We recall that $\mathcal{C}[a,b]$ is the vector space of real continuous functions on the interval $[a,b]$, and $\mathcal{C}^k[a,b]$, $k = 1, 2, \ldots$ are the subspaces of functions with k continuous derivatives, so that

$$\cdots \subset \mathcal{C}^2[a,b] \subset \mathcal{C}^1[a,b] \subset \mathcal{C}[a,b].$$

These are abstract vector spaces, and we shall now consider an inner product in them: for any $f, g \in \mathcal{C}[a,b]$,

$$(f,g) = \int_a^b f(x)g(x)\,dx.$$

Just like the Euclidean inner product $x^T y$ in $\mathbb{R}^n$, this inner product has some basic (defining) properties:

(i) $(f, g+h) = (f,g) + (f,h)$

(ii) $(f,g) = (g,f)$

(iii) $(f,f) \geq 0$, and $(f,f) = 0 \iff f = 0$

Definition: For $f \in \mathcal{C}[a,b]$ the norm is

$$\|f\| = \sqrt{(f,f)} = \sqrt{\int_a^b f(x)^2\, dx}.$$

The integrand is always greater than or equal to zero and the integral can only be 0 when the integrand is identically equal to zero, i.e. $f = 0$.

Definition: $f, g \in \mathcal{C}[a,b]$ are orthogonal if

$$(f,g) = 0.$$

Example: Let $[a,b] = [0, 2\pi]$. Then, the functions $1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots$ are orthogonal. To see this, we use the following identities:

$$\sin mx \cos nx = \tfrac{1}{2}\left(\sin(m+n)x + \sin(m-n)x\right)$$
$$\sin mx \sin nx = \tfrac{1}{2}\left(\cos(m-n)x - \cos(m+n)x\right)$$
$$\cos mx \cos nx = \tfrac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right)$$

Then, for example,

$$(\sin x, \cos 2x) = \int_0^{2\pi} \sin x \cos 2x\, dx = \tfrac{1}{2}\int_0^{2\pi} \left(\sin 3x - \sin x\right)\, dx = 0$$

A (trigonometric) Fourier Series is

$$y(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$$

We can use orthogonality to determine the coefficients. Suppose we want to find b_2. Then, we take the inner product of both sides with $\sin 2x$:

$$(y, \sin 2x) = (b_2 \sin 2x, \sin 2x),$$

that is

$$b_2 \int_0^{2\pi} \sin^2 2x \, dx = \int_0^{2\pi} y(x) \sin 2x \, dx$$

and

$$b_2 = \frac{1}{\pi} \int_0^{2\pi} y(x) \sin 2x \, dx,$$

since

$$\int_0^{2\pi} \sin^2 2x \, dx = \frac{1}{2} \int_0^{2\pi} (1 - \cos 4x) \, dx = \pi.$$

A partial sum $a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx$ of the trigonometric Fourier series is called a <u>trigonometric polynomial</u> of degree n.

What trigonometric polynomial of degree n gives the best approximation to a function $y(x)$ on $[0, 2\pi]$ in the sense of least squares, i.e. it minimizes

$$\|y - p_n\|$$

where p_n is a trigonometric polynomial of degree n

$$\alpha_0 + \alpha_1 \cos x + \beta_1 \sin x + \cdots + \alpha_n \cos nx + \beta_n \sin nx?$$

Solution: The coefficients of p_n must be the Fourier coefficients

$$\alpha_0 = a_0 = \tfrac{1}{2\pi} \int_0^{2\pi} y(x)\, dx,$$

$$\alpha_k = a_k = \tfrac{1}{\pi} \int_0^{2\pi} y(x) \cos nx\, dx, \qquad k \geq 1,$$

$$\beta_k = b_k = \tfrac{1}{\pi} \int_0^{2\pi} y(x) \sin nx\, dx, \qquad k \geq 1.$$

To see this we note that

$$E^2 = \|y - p_n\|^2 = (y, y) - 2(y, p_n) + (p_n, p_n).$$

Now,

$$(p_n, p_n) = \pi \left(2\alpha_0^2 + \alpha_1^2 + \cdots + \alpha_n^2 + \beta_1^2 + \cdots + \beta_n^2\right)$$

and $(y, p_n) =$

$$\pi \left(2\alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_n a_n + \beta_1 b_1 + \cdots + \beta_n b_n\right)$$

so that

$$E^2 = (y, y) - 2\pi \left(2\alpha_0 a_0 + \sum_{k=1}^{n} \alpha_k a_k + \beta_k b_k\right) + \pi \left(2\alpha_0^2 + \sum_{k=1}^{n} (\alpha_k^2 + \beta_k^2)\right).$$

If we use P_n, the trigonometric polynomial with the Fourier coefficients, we obtain the error

$$E^{*2} = (y, y) - (P_n, P_n) = (y, y) - \pi \left(2a_0^2 + \sum_{k=1}^{n} (a_k^2 + b_k^2)\right).$$

Thus, $E^2 - E^{*2} =$

$$2\pi(\alpha_0 - a_0)^2 + \pi \sum_{k=1}^{n} \left((\alpha_k - a_k)^2 + (\beta_k - b_k)^2\right),$$

so that $E^2 > E^{*2}$ unless $p_n = P_n$.

Example: What is the best approximation of $\cos x$ on the interval $[0, 2\pi]$ by $a \sin x + b \sin 2x$?

Solution: $a = b = 0$ since $\cos x$ is orthogonal to both $\sin x$ and $\sin 2x$ on the interval $[0, 2\pi]$.

Example: What is the best straight line approximation of $\cos x$ on $[-\pi/2, \pi/2]$?

Solution: Since 1 and x are orthogonal on that interval, we can use the projections onto those directions to easily find the best line:

$$C + Dx = \frac{(\cos x, 1)}{(1, 1)}\, 1 + \frac{(\cos x, x)}{(x, x)}\, x$$

$$= \frac{\int_{-\pi/2}^{\pi/2} \cos x \, dx}{\pi} + \frac{\int_{-\pi/2}^{\pi/2} x \cos x \, dx}{\int_{-\pi/2}^{\pi/2} x^2 \, dx}\, x$$

$$= \frac{2}{\pi} + 0x$$

Legendre Polynomials

Since it is easier to work with orthogonal polynomials, the Legendre polynomials are a simple set thus designed on the interval $[-1, 1]$. Since x is an odd function, we can start with $v_0 = 1$ and $v_1 = x$. We then use Gram-Schmidt (without normalization) on the sequence $1, x, x^2, x^3, \ldots$:

$$v_2 = x^2 - \frac{(1, x^2)}{(1, 1)} 1 - \frac{(x, x^2)}{(x, x)} x = x^2 - \frac{1}{3}.$$

The polynomials $v_0, v_1, v_2, v_3, \ldots$ generated this way are the Legendre polynomials. They are orthogonal on the interval $[-1, 1]$:

$$(v_i, v_j) = \int_{-1}^{1} v_i(x) v_j(x)\, dx = \frac{2}{2i+1} \delta_{ij}, \quad 1 \leq i, j.$$