MA 511, Session 22

More on Vector Subspaces

Let V and W be subspaces of U, and let us define their sum,

$$V + W = \{u = v + w \in U, \text{ where } v \in V, w \in W\}.$$

It is easy to see that the intersection of V and W, $V \cap W$, is a subspace of U, while, in general, their union, $V \cup W$ in not. For example, let $U = \mathbb{R}^2$, V the x-axis and W the y-axis. Then $V \cup W$ is not a subspace since $(1,0)^T$ and $(0,1)^T$ are in $V \cup W$ but their sum $(1,1)^T$ is not.

Theorem: $V \cap W$ is a subspace of U.

Proof: (i) Assume $u_1, u_2 \in V \cap W$. Then, $u_1, u_2 \in V \Rightarrow u_1 + u_2 \in V$ and $u_1, u_2 \in W \Rightarrow u_1 + u_2 \in W$. Hence, $u_1 + u_2 \in V \cap W$.

(ii) Assume $u \in V$ and $c \in \mathbb{R}$. Then, $cu \in W$ and $cu \in W$. Hence, $cu \in V \cap W$.

Example: Let $V = \mathcal{M}_{3\times 3}$ be the vector space of 3×3 matrices with real coefficients. Let L be the subspace of lower triangular matrices and U be the subspace of upper triangular matrices. Then,

$$\begin{aligned} L \cap U &= \{\text{diagonal matrices}\}, \\ L + U &= \mathcal{M}_{3\times 3}. \end{aligned}$$

Example: Let $U = \mathbb{R}^4$, V be the subspace of vectors with fourth component equal to the sum of the first two components, and W be the subspace of vectors with fourth component equal to the sum of the first and third components. Then, $V + W = \mathbb{R}^4$.

Proof: Note that

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ x+y \end{pmatrix}, \ x, y, z \in \mathbb{R} \right\}$$

$$W = \left\{ \begin{pmatrix} a \\ b \\ c \\ a+c \end{pmatrix}, \ a, b, c \in \mathbb{R} \right\}$$

and then, as the system

$$\begin{aligned} x + \qquad\quad a &= b_1 \\ y + \qquad\quad b &= b_2 \\ z + \qquad\quad c &= b_3 \\ x + y + a + c &= b_4 \end{aligned}$$

is solvable (consistent) for **any** $b \in \mathbb{R}^4$, it follows that $V + W = \mathbb{R}^4$.

Example: Let now A be a $k \times n$ matrix and B be a $l \times n$ matrix. Consider now $V = \mathcal{C}(A^T)$ and $W = \mathcal{C}(B^T)$. Then,

$$\dim(V + W) = \operatorname{rank} C,$$

where C is the $(k+l) \times n$ matrix $\begin{pmatrix} A \\ B \end{pmatrix}$.

<u>Proof</u>: By definition, the row space of C consists of all linear combinations of rows of A and rows of B, which is $V + W$ (sums of linear combinations of rows of A with linear combinations of rows of B). Also, $\dim \mathcal{C}(C^T) = \dim \mathcal{C}(C) = \operatorname{rank} C$ is the number of linearly independent rows (and columns) in C, and thus $\dim(V + W) = \operatorname{rank} C$.

Example: Similarly, let A be a $m \times k$ matrix and B be a $m \times l$ matrix. Consider $V = \mathcal{C}(A)$ and $W = \mathcal{C}(B)$. Then,

$$\dim(V + W) = \operatorname{rank} D,$$

where D is the $m \times (k+l)$ matrix $\begin{pmatrix} A & B \end{pmatrix}$.

Suppose now that V and W are subspaces of $\mathbb{R}^m$ of dimensions k and l, respectively. Let $v_1, \dots, v_k$ be a basis for V and $w_1, \dots, w_l$ be a basis for W. Let

$$D = \begin{pmatrix} v_1 & \dots & v_k & w_1 & \dots & w_l \end{pmatrix},$$

a $m \times (k + l)$ matrix. We just saw that

$$\dim(V + W) = \operatorname{rank} D.$$

Theorem: (Dimension formula)

$$\dim(V + W) + \dim(V \cap W) = \dim V + \dim W.$$

<u>Proof</u>: Let us prove that $\dim(V \cap W) = \dim \mathcal{N}(D)$.

To see this, let $x \in \mathcal{N}(D) \subset \mathbb{R}^{k+l}$. Then, $Dx = 0$ means

$$x_1 v_1 + \cdots + x_k v_k + x_{k+1} w_1 + \cdots + x_{k+l} w_l = 0,$$

that is,

$$y = x_1 v_1 + \cdots + x_k v_k = -x_{k+1} w_1 - \cdots - x_{k+l} w_l.$$

Then $y \in V \cap W$ and we have a one-to-one correspondence

$$x \in \mathcal{N}(D) \longleftrightarrow y \in V \cap W.$$

This correspondence defines linear transformations $L : \mathcal{N}(D) \to V \cap W$ and $T : V \cap W \to \mathcal{N}(D)$ such that LT and TL are identity transformations of $\mathcal{N}(D)$ and $V \cap W$, respectively. Choosing bases in $\mathcal{N}(D)$ and $V \cap W$, one can represent L and T by matrices A and B so that $AB = \mathbb{I}$ and $BA = \mathbb{I}$. Hence A and B are invertible square matrices. This shows that $\dim(V \cap W) = \dim \mathcal{N}(D)$, as needed. Finally,

$$\dim \mathcal{C}(D) + \dim \mathcal{N}(D) = k + l = \dim V + \dim W$$

completes the proof.

Example: Let V and W be as in the previous example in $\mathbb{R}^4$. Then,

$$V = \mathcal{C}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad W = \mathcal{C}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

It follows that

$$\dim(V+W) = 4 = \text{rank}\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

and $\dim(V \cap W) = \dim V + \dim W - \dim(V+W) = 2$. A basis for $V \cap W$ is, e.g. $(1,0,0,1)^T$, $(0,1,1,1)^T$.

Theorem: Let A be a $m \times n$ matrix and B be a $n \times p$ matrix. Then,

(i) $$\mathcal{N}(B) \subset \mathcal{N}(AB),$$

(ii) $$\mathcal{C}(AB) \subset \mathcal{C}(A),$$

(iii) $$\mathcal{N}(A^T) \subset \mathcal{N}\big((AB)^T\big),$$

(iv) $$\mathcal{C}\big((AB)^T\big) \subset \mathcal{C}(B^T).$$

Proof: (i) Let $x \in \mathcal{N}(B)$. Then, $Bx = 0 \Rightarrow ABx = A0 = 0$, that is $x \in \mathcal{N}(AB)$.

(ii) The columns of AB are linear combinations of the columns of A.

(iii) and (iv) are analogous, working with the transposes of A, B, AB.

Corollary: Taking dimensions, we immediately see that

(i) $\operatorname{rank}(AB) \leq \operatorname{rank} A$,

(ii) $\operatorname{rank}(AB) \leq \operatorname{rank} B$,

(iii) $\dim \mathcal{N}(AB) \geq \dim \mathcal{N}(B)$.

Definition: Given a $m \times n$ matrix A, and positive integers k, l such that $1 \leq k \leq m$ and $1 \leq l \leq n$, a $\underline{(k \times l) \text{ submatrix}}$ of A is obtained by deleting any $m - k$ rows and any $n - l$ columns of A.

Theorem: Suppose A is of rank r. Then,

(i) Every submatrix of A has rank $\leq r$.

(ii) At least one $r \times r$ submatrix of A has rank r.

Example: Find the rank and a largest invertible submatrix of $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$.

Solution: We take A to its row echelon form.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that $\operatorname{rank} A = 2$ and the largest invertible submatrix is 2×2. Any 2×2 submatrix of A is invertible, for example

$$\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}.$$