MA 511, Session 23

Determinants

Consider the vector space V of $n \times n$ matrices with real coefficients, $\mathcal{M}_{n \times n}$. We want to define a very useful function

$$\det: V \longrightarrow \mathbb{R}$$

that has the following two properties:

(1) it is linear on the first row;

(2) it is multiplied by -1 when two rows are interchanged;

(3) it takes on the value 1 on the identity matrix.

It turns out that for each positive integer n, such function is unique. An alternative notation is

$$|A| = \det A.$$

Example: Consider n = 1, that is $V = \mathbb{R}$. A "matrix" in V has one row and one column (in fact, only one coefficient). Thus, ANY element of V is of the form $c\mathbb{I}_1 = c \cdot 1 = c$.

Let now $f: V \longrightarrow \mathbb{R}$ satisfy conditions (1)–(3). Then, $f(c) = f(c\mathbb{I}_1) = cf(\mathbb{I}) = c \cdot 1 = c$, and we see that the only such function is the identity, $\det(c) = c$.

Properties of Determinants:

(1)
$$\det \begin{pmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ a_{21} & \dots & a_{2n} \\ & \ddots & & \\ & a_{n1} & \dots & a_{nn} \end{pmatrix}$$
$$= \det \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ & \ddots & \\ & a_{n1} & \dots & a_{nn} \end{pmatrix} + \det \begin{pmatrix} b_{11} & \dots & b_{1n} \\ a_{21} & \dots & a_{2n} \\ & \ddots & \\ & a_{n1} & \dots & a_{nn} \end{pmatrix}$$

and

$$\det \begin{pmatrix} ca_{11} & \dots & ca_{1n} \\ a_{21} & \dots & a_{2n} \\ & \ddots & & \\ & \ddots & & \\ & a_{n1} & \dots & a_{nn} \end{pmatrix} = c \det \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ & \ddots & & \\ & \ddots & & \\ & a_{n1} & \dots & a_{nn} \end{pmatrix}$$

(2) Let R_i denote the *i*-th row of $A, 1 \leq i \leq n$. Then,

$$\det \begin{pmatrix} R_1 \\ \dots \\ R_i \\ \dots \\ R_j \\ \dots \\ R_n \end{pmatrix} = -\det \begin{pmatrix} R_1 \\ \dots \\ R_j \\ \dots \\ R_i \\ \dots \\ R_n \end{pmatrix}$$

Note that properties (1) and (2) imply the linearity of the determinant on each row, not just the first.

(3) det $\mathbb{I} = 1$

(4) If 2 rows of A are equal, then det A = 0: this follows from (2) because, when we interchange the two equal rows, the resulting matrix is still A but its determinant is multiplied by -1. That is, det A = $-\det A \Rightarrow \det A = 0$.

(5) The elementary row operation used for elimination —that is replacing a row by itself plus a multiple of another row— does not change the determinant:

$$\det \begin{pmatrix} R_1 \\ \dots \\ R_i + cR_j \\ \dots \\ R_j \\ \dots \\ R_n \end{pmatrix} = \det \begin{pmatrix} R_1 \\ \dots \\ R_i \\ \dots \\ R_j \\ \dots \\ R_n \end{pmatrix} + c \det \begin{pmatrix} R_1 \\ \dots \\ R_j \\ \dots \\ R_j \\ \dots \\ R_n \end{pmatrix} = |A| + c \cdot 0 = |A|$$

(6) If A has a zero row, then $\det A = 0$: This follows from the linearity of $\det A$ with respect to any row. Note that A does not change if we multiply its zero row by 0.

(7) If A is triangular, then det $A = a_{11}a_{22}\ldots a_{nn}$, the product of the coefficients on the main diagonal:

Suppose first that all the coefficients on the main diagonal are nonzero. Then, we can apply elimination to produce a *diagonal* matrix with the same determinant (if the matrix is upper triangular, the elimination is done right-to-left and bottom-to-top, rather than the usual way top-to-bottom and left-toright). Then we apply linearity in each row and are led to the identity matrix multiplied by the product of all the coefficients on the diagonal. Thus, we obtain the desired formula using (3).

Finally, if one of the diagonal entries is zero, elimination will produce a zero row and, by (6), the determinant is zero.

(8) A is singular if, and only if det A = 0:

If A is singular, elimination leads to a matrix with a zero row and the same determinant as A. Thus, det A = 0. If A is nonsingular, elimination leads to an upper triangular matrix U with all diagonal entries different from zero (the pivots of A), and det $A = \pm \det U = \pm u_{11} \dots u_{nn}$, where the plus or the minus depends on the parity of the number of row exchanges that were used during elimination (plus for even, minus for odd).

(9) The determinant of a product is the product of the determinants, i.e. det(AB) = (det A)(det B):

If B is singular, so is AB and the formula holds since 0 = 0. So, we may assume B is nonsingular, and consider the function $d : \mathcal{M}_{n \times n} \longrightarrow \mathbb{R}$ defined by

$$d(A) = \frac{\det(AB)}{\det B}.$$

It is easy to see that d satisfies properties (1)–(3). Thus, $d = \det$ and the formula holds.

(10) Transposition does not change the determinant, i.e. $det(A^T) = det A$:

First, A is singular if, and only if A^T is singular, and they both have determinant equal to zero. So, assume A is nonsingular. Then, we can find a permutation matrix P, a lower triangular matrix Lwith 1's on the main diagonal, an upper triangular matrix U with 1's on the main diagonal, and a diagonal matrix D such that PA = LDU. Taking transposes, $A^T P^T = U^T D^T L^T$, where the determinants of L, U, L^T, U^T are all equal to one. The permutation matrix P is a product of elementary permutation matrices P_k , and P^T is the product of P_k in the opposite order, since $P_k^T = P_k$. Each P_k is obtained by an exchange of two rows of \mathbb{I} , hence det $P_k = -1$. This implies det $P = \det P^T = \pm 1$. Also, $D = D^T$. Hence, det $A = \frac{\det D}{\det P} = \frac{\det D^T}{\det P^T} = \det A^T$, as needed.

Example: Consider n = 2, that is $V = \mathcal{M}_{2 \times 2}$. Let us find the only function

$$\det:V\longrightarrow\mathbb{R}$$

that satisfies (1)–(3). Assume first that $a \neq 0 \in \mathbb{R}$. Then, we have

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \, \det \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix}$$

(by linearity in first row)

$$= a \, \det \left(\begin{matrix} 1 & \frac{b}{a} \\ 0 & d - \frac{c}{a} b \end{matrix} \right)$$

(by property (5))

$$= a \left(d - \frac{c}{a} b \right) \det \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$$

(by linearity in second row)

$$= (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(by property (5))

$$= ad - bc$$

(by property (3))

Suppose now a = 0. If b = 0 or c = 0, then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has a zero row or column and its determinant is zero by (6) and (9), in agreement with the formula det A = 0d - bc. Finally, for $b \neq 0 \neq c$,

$$\det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = -\det \begin{pmatrix} c & d \\ 0 & b \end{pmatrix}$$
(by property (2))

$$= -bc \det \begin{pmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{pmatrix}$$

(by linearity in each row)

$$= 0d - bc \, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(by property (5))

$$= ad - bc$$

(by property (3))

Therefore,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Example:

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$$
$$= \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Example:

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{pmatrix}$$
$$= \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= (1)(-3)(1) = -3$$