MA 511, Session 24

Formulas for Determinants

Let A be a $n \times n$ matrix. Then we may write $PA = LU$, where $L, P, U \in \mathcal{M}_{n\times n}$ are, respectively, a lower triangular matrix with 1's on the main diagonal, a permutation matrix (with determinant equal to ± 1), and an upper triangular matrix containing the pivots of the elimination on its main diagonal. Thus we are led to the formula

$$\det A = \pm \,\text{product of the pivots in } U$$

With a huge effort this observation could be used to derive a theoretical definition of $\det A$, that would be useful for the theory, but usually not for computational purposes.

Definition: A permutation of the positive integers $(1, 2, \ldots, n)$ is a one-to-one function

$$\sigma : (1, 2, \ldots, n) \longrightarrow (1, 2, \ldots, n).$$

The set of all permutations of $(1, 2, \ldots, n)$ is called the permutation group of n elements and is denoted $\mathbb{P}^n$. The number of elements (cardinality) of $\mathbb{P}^n$ is $n! = 1 \cdot 2 \ldots n$.

Example: Let $n = 3$. Then, $\mathbb{P}^3$ has six elements, $\sigma_1, \sigma_2, \ldots, \sigma_6$, respectively represented by

$$\begin{array}{ll} \sigma_1 : (1, 2, 3) & \quad \sigma_4 : (2, 3, 1) \\ \sigma_2 : (1, 3, 2) & \quad \sigma_5 : (3, 1, 2) \\ \sigma_3 : (2, 1, 3) & \quad \sigma_6 : (3, 2, 1) \end{array}$$

Thus, for example, $\sigma_5(1) = 3,\ \sigma_5(2) = 1,\ \sigma_5(3) = 2$.

Remark: There is a one-to-one correspondence between elements of $\mathbb{P}^n$ and $n \times n$ permutation matrices P: for any $\sigma \in \mathbb{P}^n$, take $P_{i\sigma(i)} = 1$, the rest of P_{ij} being 0. Conversely, given P, define σ as $P\,(1, 2, \ldots, n)^T$.

Example: Given $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, we have the uniquely associated permutation $\sigma \in \mathbb{P}^4$ represented by $(2, 4, 3, 1)$, that is, $\sigma(1) = 2,\ \sigma(2) = 4,\ \sigma(3) = 3,\ \sigma(4) = 1$.

Definition: The <u>sign</u> of a permutation $\sigma \in \mathbb{P}^n$ is $\det P_\sigma$, where P_σ is the $n \times n$ permutation matrix associated with σ. It is denoted by $\operatorname{sgn} \sigma = \det P_\sigma$.

Remark: $\operatorname{sgn} \sigma = 1$ if an even number of exchanges are needed to bring σ to the identity $(1, 2, \ldots, n)$; $\operatorname{sgn} \sigma = -1$ if an odd number of exchanges are needed to bring σ to the identity $(1, 2, \ldots, n)$.

Example: $\operatorname{sgn}(2, 3, 1) = 1$ and $\operatorname{sgn}(2, 1, 3) = -1$.

Theorem: Let $A = (a_{ij})$ be a $n \times n$ matrix. Then,

$$\det A = \sum_{\sigma \in \mathbb{P}^n} \operatorname{sgn} \sigma \ a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}$$

Example: For $n = 3$, we have

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Example:

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} = 1 \cdot 5 \cdot 10 - 1 \cdot 6 \cdot 8 - 2 \cdot 4 \cdot 10 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7 = 230 - 233 = -3$$

Definition: Given a $n \times n$ matrix A, the <u>ij-minor</u> M_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix obtained by crossing out the i-th row and the j-th column of A ($1 \le i, j \le n$).

Definition: Given a $n \times n$ matrix A, the <u>ij-cofactor</u> C_{ij} is

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The next theorem gives the cofactor method for the computation of determinants.

Theorem:

$$\text{(i)} \qquad \det A = \sum_{j=1}^{n} a_{ij}C_{ij}, \quad i \text{ fixed}, 1 \le i \le n;$$

(expansion by i-th row)

$$\text{(ii)} \qquad \det A = \sum_{i=1}^{n} a_{ij}C_{ij}, \quad j \text{ fixed}, 1 \le j \le n.$$

(expansion by j-th column)

Example: Expansion by second row.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix} = -4\begin{vmatrix} 2 & 3 \\ 8 & 10 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 10 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

$$= -4(20 - 24) + 5(10 - 21) - 6(8 - 14)$$

$$= 16 - 55 + 36 = -3$$