MA 511, Session 25

Applications of Determinants

A formula for A^{-1}

Let us define the matrix of cofactors

$$C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}$$

Theorem: $A^{-1} = \dfrac{1}{\det A}\, C^T$

Proof: Let us see that $A\,C^T = (\det A)\,\mathbb{I}$. For the diagonal entries, we know from the row expansion formula for the determinant, that

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}$$

for each index i, $1 \leq i \leq n$. As for the off-diagonal coefficients, let us prove the orthogonality of rows to vectors of cofactors,

$$(*) \qquad 0 = \sum_{j=1}^{n} a_{ij} C_{kj},$$

where $i \neq k$ are fixed, $1 \leq i, k \leq n$. This follows immediately by observing that the matrix that is obtained from A by replacing its k-th row by the i-th row has determinant equal to zero and, expanding its determinant by the k-th row, the formula (*) is obtained.

Example: Find the inverse of A using cofactors, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}.$$

Solution:

$$C_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 10 \end{vmatrix} = 2; \; C_{12} = -\begin{vmatrix} 4 & 6 \\ 7 & 10 \end{vmatrix} = 2; \; C_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3;$$

$$C_{21} = -\begin{vmatrix} 2 & 3 \\ 8 & 10 \end{vmatrix} = 4; \; C_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 10 \end{vmatrix} = -11; \; C_{23} = -\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 6;$$

$$C_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3; \; C_{32} = -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6; \; C_{33} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3.$$

Since we know that $\det A = -3$, we finally have

$$A^{-1} = -\frac{1}{3}\begin{pmatrix} 2 & 4 & -3 \\ 2 & -11 & 6 \\ -3 & 6 & -3 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{pmatrix}.$$

Cramer's Rule for the Solution of a Square Linear System

Consider the linear system $Ax = b$, where A is a nonsingular $n \times n$ matrix. Then, for $1 \leq j \leq n$,

$$x_j = \frac{\det B_j}{\det A},$$

where B_j is obtained by replacing the j-th column of A with b.

Proof: For $1 \leq j \leq n$,

$$\det B_j = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{jn},$$

which is the j-th component of the vector

$$C^T b = (\det A) A^{-1} b.$$

Example: Use Cramer's rule to solve the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + 5x_2 + 6x_3 &= 2 \\ 7x_1 + 8x_2 + 10x_3 &= 0 \end{aligned}$$

<ins>Solution</ins>: We know that $\det A = -3$, where A is the coefficient matrix for the system. Now we compute

$$\det B_1 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 0 & 8 & 10 \end{vmatrix} = (50 - 48) - 2(20 - 24) = 10$$

$$\det B_2 = \begin{vmatrix} 1 & 1 & 3 \\ 4 & 2 & 6 \\ 7 & 0 & 10 \end{vmatrix} = -(40 - 42) + 2(10 - 21) = -20$$

$$\det B_3 = \begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \\ 7 & 8 & 0 \end{vmatrix} = (32 - 35) - 2(8 - 14) = 9$$

and thus, $x_1 = -\frac{10}{3}$, $x_2 = \frac{20}{3}$, $x_3 = -3$.