MA 511, Session 26

Review

Example: Find a unit vector v orthogonal to the row space of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Solution: Since the null space is the orthogonal complement of the row space, any vector orthogonal to the latter lies in the former. Thus, we seek a nonzero vector x such that $Ax = 0$. We use elimination:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} = U,$$

and we solve $Ux = 0$. We find that $x = t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is the null space of A. It contains exactly two vectors of unit length: $v = \pm \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Let $v_1, \ldots, v_n \in \mathbb{R}^m$ span a subspace W of $\mathbb{R}^m$, and let $b \in \mathbb{R}^m$. To find the vector $v \in W$ that minimizes the distance $\|b - v\|$ we take v as the (orthogonal) projection of b into W.

If $v_1, \ldots, v_n$ are orthogonal, this is done very easily:

$$v = \frac{(b, v_1)}{(v_1, v_1)} v_1 + \frac{(b, v_2)}{(v_2, v_2)} v_2 + \cdots + \frac{(b, v_n)}{(v_n, v_n)} v_n.$$

Example: The vector in the xy-plane of $\mathbb{R}^3$ closest to $(1, 2, 3)$ is

$$\begin{aligned} v = (1 \quad 0 \quad 0) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (0 \quad 1 \quad 0) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}. \end{aligned}$$

Here we view the xy-plane as the subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (orthonormal).

Example: Find the function $A\sin x + B\sin 2x + C\cos 2x$ closest to $\sin^2 x$ in the sense of least squares on the interval $[-\pi, \pi]$.

Solution: Since $\sin x, \sin 2x, \cos 2x$ are orthogonal on the interval $[-\pi, \pi]$, the solution is just the sum of the projections in the three directions given by these functions:

$$f(x) = \frac{\int_{-\pi}^{\pi} \sin x \sin^2 x \, dx}{\int_{-\pi}^{\pi} \sin^2 x \, dx} \sin x$$

$$+ \frac{\int_{-\pi}^{\pi} \sin 2x \sin^2 x \, dx}{\int_{-\pi}^{\pi} \sin^2 2x \, dx} \sin 2x$$

$$+ \frac{\int_{-\pi}^{\pi} \cos 2x \sin^2 x \, dx}{\int_{-\pi}^{\pi} \cos^2 2x \, dx} \cos 2x$$

$$= 0 + 0 + \frac{-\frac{\pi}{2}}{\pi} \cos 2x = -\frac{1}{2} \cos 2x,$$

since

$$\begin{cases} \int_{-\pi}^{\pi} \cos^2 2x \, dx = \pi, \\ \int_{-\pi}^{\pi} \cos 2x \sin^2 x \, dx = \int_{-\pi}^{\pi} \cos 2x \big[\frac{1}{2}(1 - \cos 2x)\big] \, dx = -\frac{\pi}{2}. \end{cases}$$

Example: If $v_1, \ldots, v_n$ are not orthogonal in $\mathbb{R}^m$, then the closest vector to $b \in \mathbb{R}^m$ in their span W is still the orthogonal projection, $p = A\bar{x}$, where A is the $m \times n$ matrix $(\, v_1, \ldots, v_n \,)$, and $\bar{x}$ is the solution of the normalized system $A^T A\bar{x} = A^T b$. If $v_1, \ldots, v_n$ are linearly independent, then A has rank n and $A^T A$ is invertible. In this case we have

$$\bar{x} = (A^T A)^{-1} A^T b.$$

The matrix $P = A(A^T A)^{-1} A^T$ is, therefore, the projection matrix onto the subspace of $W \subset \mathbb{R}^m$ spanned by $v_1, \ldots, v_n$.

Example: Find the best straight line approximation $y = C + Dt$ to the data

$$\begin{aligned} y &= b_1 \qquad && \text{when} \;\; t = t_1 \\ y &= b_2 \qquad && \text{when} \;\; t = t_2 \\ &\cdots\cdots \\ y &= b_m \qquad && \text{when} \;\; t = t_m \end{aligned}$$

Typically this $m \times 2$ system has no solution and we find the least squares approximation. In matrix form, the system is $A \begin{pmatrix} C \\ D \end{pmatrix} = b$, where

$$A = \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \cdot & \\ \cdot & \\ \cdot & \\ 1 & t_m \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix}.$$

We solve the normalized system $A^T A \begin{pmatrix} \bar{C} \\ \bar{D} \end{pmatrix} = A^T b$, which is of dimension just 2×2:

$$\begin{pmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{pmatrix} \begin{pmatrix} \bar{C} \\ \bar{D} \end{pmatrix} = \begin{pmatrix} \sum b_i \\ \sum b_i t_i \end{pmatrix}.$$

Example: Find the matrix that represents the orthogonal projection of $\mathbb{R}^3$ onto the line of direction $(\,1 \quad 0 \quad 2\,)^T$.

<u>Solution</u>: Since $A^T A = (\,5\,)$,

$$P = \frac{AA^T}{5} = \frac{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} (\,1 \quad 0 \quad 2\,)}{5} = \frac{1}{5} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{pmatrix}.$$

Example: We recall that, given any $m \times n$ matrix A, $\mathcal{C}(A^T)$ and $\mathcal{N}(A)$ are orthogonal complements in $\mathbb{R}^n = \mathcal{C}(A^T) \oplus \mathcal{N}(A)$ (this means that $\mathbb{R}^n = \mathcal{C}(A^T) + \mathcal{N}(A)$ and $\mathcal{C}(A^T) \cap \mathcal{N}(A) = \{0\}$). Therefore, given any $x \in \mathbb{R}^n$, it can be uniquely written as $x = x_n + x_r$, where $x_n \in \mathcal{N}(A)$ and $x_r \in \mathcal{C}(A^T)$.

Example: Given $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$, $x = (\,3 \quad 3 \quad 3\,)^T$, write $x = x_n + x_r$, with $x_n \in \mathcal{N}(A)$, $x_r \in \mathcal{C}(A^T)$.

Solution: There are several ways to do this. One is to find the orthogonal projection of x into the row space of A, $\mathcal{C}(A^T)$. So, let us find the solution of the normalized system $AA^T\bar{y} = Ax$. Then, $x_r = A^T\bar{y}$. We have

$$AA^T = \begin{pmatrix} 5 & 9 \\ 9 & 18 \end{pmatrix}, \qquad Ax = \begin{pmatrix} 9 \\ 18 \end{pmatrix}.$$

We solve the system

$$\begin{aligned} 5\bar{y}_1 + 9\bar{y}_2 &= 9 \\ 9\bar{y}_1 + 18\bar{y}_2 &= 18 \end{aligned} \qquad\qquad \begin{aligned} 5\bar{y}_1 + 9\bar{y}_2 &= 9 \\ -\bar{y}_1 &= 0 \end{aligned}$$

and we see that

$$\bar{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; x_r = A^T \bar{y} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix},$$

and so,

$$x_n = x - x_r = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

Alternatively, we can find x_n as a projection of x into the nullspace of A, and $x_r = x - x_n$. Since $\mathcal{N}(A)$ is spanned by $z = (\,-2 \quad -2 \quad 1\,)^T$,

$$x_n = \frac{(x, z)}{(z, z)} z = \frac{-9}{9} \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

Example: Use the Gram-Schmidt orthonormalization method to find three functions f_1, f_2, f_3 which are linear combinations of x, x^4, x^5 and are orthonormal on the interval $[-1, 1]$.

Solution: Since x and x^4 are already orthogonal on $[-1, 1]$, we take

$$f_1(x) = \sqrt{\frac{3}{2}}\, x, \qquad f_2 = \frac{3}{\sqrt{2}}\, x^4.$$

Let now

$$\begin{aligned} v_3(x) &= x^5 - (f_1, x^5) f_1 - (f_2, x^5) f_2 \\ &= x^5 - \left(\sqrt{\frac{3}{2}} \int_{-1}^{1} x^6\, dx \right) \sqrt{\frac{3}{2}}\, x \\ &\quad - \left(\frac{3}{\sqrt{2}} \int_{-1}^{1} x^9\, dx \right) \frac{3}{\sqrt{2}}\, x^4 \\ &= x^5 - (\tfrac{3}{2} \cdot \tfrac{2}{7})\, x \\ &= x^5 - \tfrac{3}{7}\, x \end{aligned}$$

and

$$
\begin{aligned}
f_3(x) &= \frac{1}{\sqrt{\int_{-1}^{1}(x^5 - \frac{3}{7}x)^2 dx}}\,(x^5 - \tfrac{3}{7}\,x) \\
&= \frac{1}{(\frac{2}{11} - \frac{6}{49})^{1/2}}\,(x^5 - \tfrac{3}{7}\,x) \\
&= \frac{\sqrt{22}}{8}\,(7x^5 - 3x)
\end{aligned}
$$