MA 511, Session 27

Eigenvalues and Eigenvectors

Let A be a $n \times n$ matrix.

Definition: We say the number λ (real or complex) is an eigenvalue of A if $\det(A - \lambda\mathbb{I}) = 0$.

Remark: $p(\lambda) = \det(A - \lambda\mathbb{I})$ is a polynomial of degree n, the characteristic polynomial of A. The eigenvalues of A are the roots of p. The total number of eigenvalues, counted with multiplicities, is n. We also remark that when A is real, if $\lambda = a + bi$ is an eigenvalue, then so is its complex conjugate $\bar{\lambda} = a - bi$.

$$p(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}$$

$$= \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \dots + \alpha_0$$

$$= (-1)^n \lambda^n + (-1)^{n-1}(\operatorname{tr} A)\lambda^{n-1} + \dots + \det A,$$

where $\operatorname{tr} A = a_{11} + a_{22} + \dots + a_{nn}$ is the trace of A.

To see that these three coefficients are indeed of the given form, recall the formula

$$\det B = \sum_{\sigma \in \mathcal{P}^n} (\operatorname{sgn} \sigma)\, b_{1\sigma(1)} b_{2\sigma(2)} \dots b_{n\sigma(n)},$$

and apply it to $B = A - \lambda\mathbb{I}$. We see that the only permutation σ that will result in terms of the form λ^n and λ^{n-1} is the identity permutation, which gives the term $(a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda)$. Collecting the terms in this product with λ^n and λ^{n-1}, we see that they are of the form claimed. As for the constant term, simply substitute $\lambda = 0$.

Remark: If λ is an eigenvalue of A, then $A - \lambda\mathbb{I}$ is singular, so that

$$(*) \qquad (A - \lambda\mathbb{I})\, x = 0$$

has nontrivial solutions.

Definition: A nontrivial solution of $(*)$ is called an eigenvector of A corresponding to the eigenvalue λ. The set of all solutions of $(*)$, i.e. the null space of $A - \lambda\mathbb{I}$, is the eigenspace corresponding to λ, denoted by S_λ.

Example: Given

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

find the eigenvalues and eigenvectors.

<u>Solution</u>: The three matrices have the same characteristic polynomial $p(\lambda) = (1-\lambda)^3$ with the single root $\lambda = 1$ with multiplicity 3.

For A, $A - \mathbb{I} = 0$ says all nonzero vectors are eigenvectors. Thus, $S_1 = \mathbb{R}^3$.

For B,

$$B - \mathbb{I} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

leads to the eigenspace $S_1 = \{x \in \mathbb{R}^3 : x_2 = 0\}$, which has dimension 2 (a plane). A basis is, e.g. $(1,0,0)^T$, $(0,0,1)^T$.

For C,

$$C - \mathbb{I} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

leads to the eigenspace of dimension 1 (a line) $S_1 = \{x \in \mathbb{R}^3 : x_2 = x_3 = 0\}$. A basis is, e.g. $(1,0,0)^T$.

First-Order Linear Homogeneous Systems of Ordinary Differential Equations:

$$\begin{cases} \dfrac{du_1}{dt} = a_{11}u_1 + \cdots + a_{1n}u_n \\ \qquad . \\ \qquad . \\ \qquad . \\ \dfrac{du_n}{dt} = a_{n1}u_1 + \cdots + a_{nn}u_n \end{cases}$$

We are interested in nontrivial (nonzero) solutions. We let $u(t) = \big(u_1(t), \ldots, u_n(t)\big)^T$ and write the system in matrix form as

$$\frac{du}{dt} = Au.$$

By analogy with the single equation, we seek exponential solutions $u = e^{\lambda t}v$, where $v \in \mathbb{R}^n$ is a constant vector. Substituting in the equation, we obtain

$$\lambda e^{\lambda t}v = Ae^{\lambda t}v.$$

Multiplying both sides by $e^{-\lambda t}$, we see that $Av = \lambda v$, that is

$$(A - \lambda \mathbb{I})\, v = 0.$$

This means that λ must be an eigenvalue of A, and v a corresponding eigenvector.

Note that, if $e^{\lambda_1 t} v_1, e^{\lambda_2 t} v_2, \ldots, e^{\lambda_k t} v_k$ are solutions, then so is any linear combination of them

$$v = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_k e^{\lambda_k t} v_k,$$

where $c_1, \ldots, c_k$ are arbitrary numbers.

Also, if $u(t) = e^{\lambda t} v$, then $u(0) = v$, so that solving the initial value problem for this system means finding the constants $c_1, \ldots, c_k$.

Example: Solve the initial value problem

$$\frac{du}{dt} = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix} u, \qquad u(0) = \begin{pmatrix} 6 \\ 3 \end{pmatrix}.$$

<u>Solution</u>: The characteristic polynomial is $p(\lambda) = \begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2$. Thus, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$.

We find S_{-1} by solving the system

$$\begin{pmatrix} 5 & -5 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A basis for S_{-1} is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Similarly, we find S_2 by solving the system

$$\begin{pmatrix} 2 & -5 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A basis for S_2 is $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

So, the general solution of the system is

$$u(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 5 \\ 2 \end{pmatrix},$$

and the initial conditions yield the system of linear equations

$$\begin{pmatrix} 6 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 5 \\ 2 \end{pmatrix},$$

which has the unique solution $c_1 = c_2 = 1$. Thus, the solution of the given initial value problem is

$$u(t) = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{2t} \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$