MA 511, Session 29

Linear Systems of ODEs and Matrix Exponentials

Let A be a $n \times n$ matrix. We want to solve the linear, homogeneous system of ODEs with constant coefficients

$$\frac{dx}{dt} = Ax.$$

Assume A is diagonalizable, $S^{-1}AS = \Lambda$. We want to show that a simple change of variables makes the system decoupled.

Define

$$y = S^{-1}x, \qquad \text{i.e.} \qquad x = Sy.$$

Then the system becomes

$$S\frac{dy}{dt} = ASx, \qquad \text{i.e.} \qquad \frac{dy}{dt} = S^{-1}ASy = \Lambda y.$$

Thus,

$$\frac{dy_j}{dt} = \lambda_j y_j, \quad \text{for} \;\; 1 \le j \le n.$$

The solutions are $y_j = c_j e^{\lambda_j t}$, $1 \leq j \leq n$. In vector form,

$$y(t) = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} + \cdots + c_n e^{\lambda_n t} \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_1 t} & \dots & 0 \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{pmatrix}$$

We see that the initial value problem $\frac{dx}{dt} = Ax, \;\; x(0) = x_0$ becomes $\frac{dy}{dt} = \Lambda y, \;\;\; y(0) = y^0 = S^{-1}x_0$, and the solution is

$$y(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ & & \cdot & \\ & & \cdot & \\ & & \cdot & \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} y_1^0 \\ y_2^0 \\ \cdot \\ \cdot \\ \cdot \\ y_n^0 \end{pmatrix},$$

where

$$\begin{pmatrix} y_1^0 \\ y_2^0 \\ \cdot \\ \cdot \\ \cdot \\ y_n^0 \end{pmatrix} = S^{-1} x_0.$$

Another approach to solving this system is using matrix exponentials.

Recall that, for any number x,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Definition: Given a $n \times n$ matrix A, the <u>matrix exponential</u> for A is

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \mathbb{I} + A + \frac{A^2}{2} + \dots.$$

If t is a scalar variable, then

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \mathbb{I} + At + \frac{A^2 t^2}{2} + \dots.$$

Assume now that S diagonalizes A, i.e. $A = S\Lambda S^{-1}$. Then

$$\begin{aligned} e^{At} &= \mathbb{I} + S\Lambda t S^{-1} + \frac{S\Lambda t^2 S^{-1}}{2} + \dots \\ &= S(\mathbb{I} + \Lambda t + \frac{\Lambda^2 t^2}{2} + \dots)S^{-1} \\ &= S e^{\Lambda t} S^{-1}. \end{aligned}$$

Note that, for a diagonal matrix Λ, we have

$$\Lambda^k = \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ & & . & \\ & & . & \\ & & . & \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix}$$

and thus

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ & & . & \\ & & . & \\ & & . & \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}.$$

Example: Given $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, find e^{At}.

<u>Solution</u>: We find the characteristic polynomial of A: $p(\lambda) = (-2-\lambda)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1)$. The eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = -1$. Since A is 2×2 and it has two distinct eigenvalues, we know that the dimension of each eigenspace is exactly 1 (a line).

For $\lambda = -3$, we solve $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and find that $S_{-3} = \text{ span }\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$.

For $\lambda = -1$, we solve $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and find that $S_{-1} = \text{ span }\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

Then, we set

$$S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}, \quad S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and

$$e^{At} = S\begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{-t} \end{pmatrix}S^{-1} = \frac{1}{2}\begin{pmatrix} e^{-3t} + e^{-t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-3t} + e^{-t} \end{pmatrix}.$$

Properties of e^A:

(i) If A is the zero matrix, then $e^A = e^0 = \mathbb{I}$:

$$e^0 = \mathbb{I} + 0 + \frac{0^2}{2} + \cdots = \mathbb{I}.$$

(ii) $e^A e^{-A} = \mathbb{I}$: Note that

$$\begin{aligned} e^A e^{-A} &= \left(\mathbb{I} + A + \frac{A^2}{2} + \frac{A^3}{6} + \dots\right)\left(\mathbb{I} - A + \frac{A^2}{2} - \frac{A^3}{6} + \dots\right) \\ &= \mathbb{I}, \end{aligned}$$

since A and $-A$ commute (all the infinitely many terms in the product cancel when collecting like terms, except for the first one, just as when we do the product with numbers rather than matrices).

Definition: A <u>fundamental matrix</u> X for the system $\frac{dx}{dt} = Ax$ is a $n \times n$ matrix whose columns are linearly independent solutions to the system.

Remark: If X is a fundamental matrix for the system $\frac{dx}{dt} = Ax$, then X itself satisfies the system:

$$\frac{dX}{dt} = AX.$$

The general solution is then Xc, where $c = \begin{pmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{pmatrix}$.

Theorem: e^{At} is a fundamental matrix for $\frac{dx}{dt} = Ax$.

<u>Proof</u>: First note that e^{At} is always invertible, thanks to property (ii). Thus, its columns are linearly independent. Next,

$$\begin{aligned} \frac{d}{dt}\, e^{At} &= \frac{d}{dt}\,(\mathbb{I} + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \ldots) \\ &= A + At^2 + \frac{A^3t^2}{2} + \frac{A^4t^3}{6} + \ldots) \\ &= A(\mathbb{I} + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \ldots) \\ &= Ae^{At}, \end{aligned}$$

as needed.

Remark: Note that, since $e^{A0} = \mathbb{I}$, the solution of the initial value problem $\frac{du}{dt} = Au, \quad u(0) = u_0$, is $u(t) = e^{At}u_0$.

Remark: A n-th order linear ODE with constant coefficients, $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0$, can be rewritten as a system by introducing n new unknowns: With

$$u_1 = y,\ u_2 = y',\ u_3 = y'',\ \ldots, u_n = y^{(n-1)},$$

we have

$$\begin{cases} \dfrac{du_1}{dt} = u_2 \\ \dfrac{du_2}{dt} = u_3 \\ \quad\vdots \\ \dfrac{du_{n-1}}{dt} = u_n \\ \dfrac{du_n}{dt} = -\dfrac{a_{n-1}}{a_n} u_n - \dfrac{a_{n-2}}{a_n} u_{n-1} - \cdots - \dfrac{a_0}{a_n} u_1. \end{cases}$$

In matrix form,

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ 0 & 0 & 0 & \ldots & 1 \\ -\frac{a_0}{a_n} & & & \ldots & -\frac{a_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix}.$$