MA 511, Session 30

<u>Stability of Solutions of Linear Systems of ODEs</u>

Let A be a $n \times n$ matrix. Then e^{At} is a fundamental matrix for $\frac{du}{dt} = Au$. Any solution of this system has the form

$$u(t) = e^{At}c.$$

If A is diagonalizable, the components of any solution are linear combinations of exponentials $e^{\lambda_k t}$, where λ_k is an eigenvalue of A.

Definition: The system is <u>stable</u> if $\Re\,\lambda < 0$ for all eigenvalues λ of A. In this case all solutions of the system decay to 0 as $t \to \infty$. (Usually this case is called asymptotically stable)

Definition: The system is <u>neutrally stable</u> if $\Re\,\lambda \leq 0$ for all eigenvalues λ of A, but $\Re\,\lambda = 0$ for some eigenvalue λ.

Definition: The system is <u>unstable</u> if $\Re\,\lambda > 0$ for some eigenvalue λ of A. In this case, some solutions of the system are unbounded.

Example: Consider the system $\frac{du}{dt} = Au$, where $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.

Then, the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$ and the system is stable.

The general solution is

$$u(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which all tend to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$.

In physics these equations represent a flow on the xy-plane. If we let $u(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ represent the coordinates of a particle at time t, then $\frac{du}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$ is the velocity vector at time t. The graph of any solution of the system from the parametric equations $x(t) = c_1 e^{-3t} + c_2 e^{-t}, \quad y(t) = c_1 e^{-3t} - c_2 e^{-t}$ is called a <u>trajectory</u> of the system. In this example, all trajectories point to the origin and, in fact, converge to it as $t \to \infty$.

Example: Consider the system $\frac{du}{dt} = Au$, where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Then, the eigenvalues are $\lambda = \pm i$ and the system is neutrally stable.

We now look for eigenvectors for $\lambda = i$. We solve the system $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and easily find that $S_i = \text{ span } \left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$.

Similarly, to find the eigenvectors for $\lambda = i$, we solve the system $\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and easily find that $S_{-i} = \text{ span } \left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} \right\}$.

Thus, the general solution of the system is

$$u(t) = c_1 e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_1 e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

To write the general real-function solution, we note that the real and the imaginary parts of solutions are themselves solutions. Let us determine them:

$$\begin{aligned}
e^{it}\begin{pmatrix}1\\-i\end{pmatrix} &= (\cos t + i\sin t)\begin{pmatrix}1\\-i\end{pmatrix}\\
&= \cos t\begin{pmatrix}1\\0\end{pmatrix} + \sin t\begin{pmatrix}0\\1\end{pmatrix} + i\left(\sin t\begin{pmatrix}1\\0\end{pmatrix} - \cos t\begin{pmatrix}0\\1\end{pmatrix}\right)\\
&= \begin{pmatrix}\cos t\\ \sin t\end{pmatrix} + i\begin{pmatrix}\sin t\\ -\cos t\end{pmatrix}
\end{aligned}$$

Thus, $\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$ and $\begin{pmatrix}\sin t\\ -\cos t\end{pmatrix}$ are solutions of the system, and the general (real) solution is

$$u(t) = c_1\begin{pmatrix}\cos t\\ \sin t\end{pmatrix} + c_2\begin{pmatrix}\sin t\\ -\cos t\end{pmatrix}.$$

We see that all these solutions are periodic, with period 2π. The trajectories are circles centered at the origin, called <u>orbits</u>.

For plane flows we have $p(\lambda) = \lambda^2 - \operatorname{tr} A\,\lambda + \det A$,

$$\lambda = \frac{\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}}{2}$$

and so, we may summarize the stability properties of the system in a <u>trace-determinant</u> diagram that represents the parabola $x = 4y^2$ in the xy-plane, with $x = \operatorname{tr} A$ and $y = \det A$. Then,

I) In the first quadrant, both λ are real positive below the parabola and complex conjugates with positive real part above the parabola, so that in both cases the system is unstable. Similarly, on the parabola in the first quadrant, there is a double, real, positive eigenvalue and the system is still unstable.

II) In the second quadrant the system is always stable. Both λ are real negative below the parabola, they coincide and are negative on the parabola, and they are complex conjugates with negative real part above the parabola.

III) In the third and fourth quadrants the eigenvalues are real, one positive and one negative, and thus the system is unstable.

IV) On the positive x-axis, one eigenvalue is 0 and the other positive so that the system is unstable, while on the negative x-axis, one eigenvalue is 0 and the other negative so that the system is neutrally stable.

V) On the positive y-axis the eigenvalues are pure imaginary, so that the system is neutrally stable.

Remark: We see that, if the system is stable, then necessarily $\operatorname{tr} A < 0, \ \det A > 0$.

Example: Consider the system $\frac{du}{dt} = Au$, where $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

Then, the characteristic polynomial is $p(\lambda) = (1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 5$ and the eigenvalues are $\lambda = 1 \pm 2i$ and the system is unstable.

We now look for eigenvectors for $\lambda = 1 - 2i$. We solve the system $\begin{pmatrix} 2i & 2 \\ -2 & 2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and easily find that $S_{1-2i} = \text{ span } \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\}$.

Thus, a complex solution of the system is

$$u(t) = e^{(1-2i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} = e^t (\cos 2t - i \sin 2t) \begin{pmatrix} i \\ 1 \end{pmatrix},$$

that gives the two real solutions

$$u_1(t) = e^t \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}, \qquad u_2(t) = e^t \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix}.$$

We see that the trajectories spiral towards the origin (counterclockwise) as $t \to -\infty$ and they spiral outward (clockwise) unboundedly as $t \to \infty$.

Example: Let us consider the second order equation that corresponds to the dynamics of a mass-spring system that undergoes damped harmonic motion. Let L_0 be the natural length of the spring and x represent the displacement from equilibrium along a horizontal line (so that gravity effects are not present). Then, the forces acting on the mass are $F_s = -kx$ and $F_f = -c\frac{dx}{dt}$, respectively the reactive force of the spring and the damping (dissipative) force due to friction, where $k > 0$ is the elastic constant of the spring and $c > 0$ the damping constant. It follows from Newton's law that the sum of all forces must equal $m\frac{d^2x}{dt^2}$, where m is the mass attached to the spring. Thus,

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

and $x(0) = x_0$ and $x'(0) = v_0$ represent, respectively, the initial displacement and initial velocity imparted to start the motion.

As a system, $u_1 = x$ and $u_2 = \frac{dx}{dt} = x'$, and

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = -\frac{c}{m}\,u_2 - \frac{k}{m}\,u_1.$$

In matrix form

$$\frac{du}{dt} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} u.$$

We see that $\det A = \frac{k}{m} \geq 0$ and $\operatorname{tr} A = -\frac{c}{m} \leq 0$. Thus the system is stable if $c > 0$ (damped harmonic motion) and neutrally stable if $c = 0$ (undamped harmonic motion).