# MA 511, Session 31

## Complex Matrices

Let us recall that the conjugate of the complex number $z = a + bi$ is the complex number $\bar{z} = a - bi$, and its modulus or absolute value is the non-negative real number $|z| = \sqrt{a^2 + b^2}$.

A number of the form $\frac{\alpha + i\beta}{\gamma + i\delta}$ can be put in the form $a + bi$ by multiplying and dividing by $\gamma - i\delta$:

$$\frac{\alpha + i\beta}{\gamma + i\delta} \frac{\gamma - i\delta}{\gamma - i\delta} = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} + i \frac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2}.$$

The inner product for vectors $x, y \in \mathbb{C}^n$ is

$$(x, y) = \bar{x}^T y.$$

**Example:** Given $x = \begin{pmatrix} 1 + i \\ 2 \end{pmatrix}, \quad y = \begin{pmatrix} i \\ 1 + i \end{pmatrix}$, find $(x, y)$.

Solution:

$$(x, y) = \bar{x}^T y = (1 - i)(i) + (2)(1 + i)$$
$$= i + 1 + 2 + 2i = 3 + 3i.$$

**Definition:** The norm or length of a complex vector $x \in \mathbb{C}^n$ is

$$\|x\| = \sqrt{(x, x)} = \sqrt{\bar{x}^T x}.$$

**Remark:** Note that, whether $x$ is a real or complex vector,

$$\|x\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}.$$

**Definition:** Given a $m \times n$ complex matrix $A$, its Hermitian transpose is the $n \times m$ matrix

$$A^H = \overline{A^T} = (\bar{A})^T.$$

**Example:** Given $A = \begin{pmatrix} 1+i & 2 \\ 3+i & 1+2i \end{pmatrix}$, we have $A^H = \begin{pmatrix} 1-i & 3-i \\ 2 & 1-2i \end{pmatrix}$.

**Remark:** The inner product in $\mathbb{C}^n$ can be written $(x, y) = x^H y$. Also, Hermitian transposition satisfies $(AB)^H = B^H A^H$ whenever the product $AB$ exists.

**Definition:** A $n \times n$ matrix $A$ is Hermitian if $A^H = A$. It is skew-Hermitian if $A^H = -A$.

**Remark:** A real matrix $A$ is Hermitian when it is symmetric.

**Theorem:** If $A$ is a $n \times n$ Hermitian matrix, then $x^H A x$ is real for all $x \in \mathbb{C}^n$.

Proof: $(x^H A x)^H = x^H A^H (x^H)^H = x^H A x$. That is, the number $x^H A x$ is equal to its conjugate. Thus, it is real.

**Theorem:** If $A$ is a $n \times n$ Hermitian matrix, then all its eigenvalues are real. In particular, all real symmetric matrices have real eigenvalues.

Proof: Let $\lambda$ be an eigenvalue of $A$ and $x$ a corresponding eigenvector. Then, $Ax = \lambda x$ and $x^H A x = x^H \lambda x = \lambda \|x\|^2$. Since the left side is real and so is $\|x\|^2$, it follows that $\lambda$ is real too.

**Theorem:** Let $\lambda_1$ and $\lambda_2$ be distinct eigenvalues of a Hermitian matrix $A$, and let $x_1$ and $x_2$ be corresponding eigenvectors. Then, $x_1^H x_2 = 0$ (i.e. eigenvectors corresponding to different eigenvalues of a Hermitian matrix are orthogonal)

Proof: Note that, since $\lambda_1$ is real,

$\lambda_1(x_1^H x_2) = (\lambda_1 x_1)^H x_2 = (A x_1)^H x_2 = x_1^H A^H x_2 = x_1^H \lambda_2 x_2 = \lambda_2(x_1^H x_2)$. Thus, $(\lambda_1 - \lambda_2)(x_1^H x_2) = 0$ implies that $x_1^H x_2 = 0$, as needed.

**Definition:** Given a $n \times n$ complex matrix $U$, we say $U$ is unitary if its columns are orthonormal in $\mathbb{C}^n$. This is equivalent to saying $U^H = U^{-1}$ (i.e. its inverse is its Hermitian transpose).

**Remark:** A real matrix $U$ is unitary if, and only if it is orthogonal.

**Theorem:** (Spectral Theorem) If $A$ is a $n \times n$ Hermitian matrix, then $A$ is diagonalizable by a unitary matrix $U$, i.e. $U^H A U = \Lambda$ is diagonal. In this case we say that $A$ is unitarily similar to a diagonal matrix. In particular, all real symmetric matrices are diagonalizable (in fact, by orthogonal matrices).

Proof: We only give the proof here for matrices that have distinct eigenvalues. These have a full set of eigenvectors and, by the last theorem, these eigenvectors are orthogonal.

**Example:** Diagonalize $A = \begin{pmatrix} 7 & -16 & -8 \\ -16 & 7 & 8 \\ -8 & 8 & -5 \end{pmatrix}$ by an orthogonal matrix $Q$.

<u>Solution</u>: The characteristic polynomial is

$$p(\lambda) = -(\lambda^3 - 9\lambda^2 - 405\lambda - 2187)$$

with roots $\lambda_1 = \lambda_2 = -9$ and $\lambda_3 = 27$. We can easily find that $S_{-9} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \right\}$ and $S_{27} = \text{span}\left\{ \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right\}$. Since these three vectors are orthogonal, we may normalize them and put them as columns of $Q = \begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} \end{pmatrix}$. Then, $Q^T = Q^{-1}$ and we see that indeed

$$Q^T A Q = \begin{pmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 27 \end{pmatrix}.$$

**Theorem:** If $U$ is unitary, then

$$x^H y = (Ux)^H (Uy) \text{ for all } x, y \in \mathbb{C}^n.$$

In particular, $\|x\| = \|Ux\|$.

Proof: $(Ux)^H (Uy) = x^H U^H U y = x^H \mathbb{I} y = x^H y$.

**Theorem:** If $\lambda$ is an eigenvalue for a unitary matrix $U$, then $|\lambda| = 1$.

Proof: Let $x$ be an eigenvector of $U$ corresponding to $\lambda$. Then,

$$|\lambda|^2 (x^H x) = (\lambda x)^H (\lambda x) = (Ux)^H (Ux) = x^H x$$

and necessarily $|\lambda| = 1$.

**Theorem:** If $U$ is unitary, then it is unitarily similar to a diagonal matrix.

**Questions:** (i) Can a Hermitian matrix be singular? Yes, 0 is Hermitian.

(ii) Can a unitary matrix be singular? No, $U^H = U^{-1}$.

(iii) Can a real symmetric matrix have non-real eigenvalues? No (theorem).

(iv) Can an orthogonal matrix have non-real eigenvalues? Yes, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ does.

(v) What is $\det U$ for unitary $U$? $\det U = z \in \mathbb{C}$ where $|z| = 1$, since it is the product of the eigenvalues of $U$ —all of which lie on the unit circle.

**Example:** If $F_n$ is the $n \times n$ Fourier matrix, then $\frac{1}{\sqrt{n}} F_n$ is unitary.