MA 511, Session 33

Jordan Forms

Let A be a $n \times n$ matrix. If A is diagonalizable, it is useful to make a change of coordinates using the similarity transformation $S^{-1}AS = \Lambda$ diagonal.

If this in not possible, then we try find a useful substitute. We saw in session 32 that for any $n \times n$ matrix A, there is a unitary U such that $U^H AU = T$ is upper triangular.

A different approach leads to Jordan canonical forms. These are built up from Jordan blocks. A Jordan block is a matrix of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

A Jordan block for a matrix A has one eigenvalue λ of A and the associated eigenspace has dimension 1.

Different blocks for the same matrix may correspond to the same eigenvalue of A.

Example: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ have the same characteristic polynomial and eigenvalues ($\lambda_1 = \lambda_2 = 1$). However, A has $\dim S_1 = 1$ while B has $\dim S_1 = 2$. Thus, B has <u>two</u> Jordan blocks of size 1, while A has <u>one</u> Jordan block of size 2.

Theorem: (Jordan's Theorem) Given a $n \times n$ matrix A with exactly s linearly independent eigenvectors $x_1, \ldots, x_s$, there exists an invertible matrix M such that

$$M^{-1}AM = J = \begin{pmatrix} J_1 & 0 & \ldots & 0 \\ 0 & J_2 & \ldots & 0 \\ & & \vdots & \\ 0 & 0 & \ldots & J_s \end{pmatrix}$$

where J_i is a Jordan block with the eigenvalue λ_i of A on the diagonal (corresponding to the eigenvector x_i), $1 \leq i \leq s$. J is called a <u>Jordan canonical form</u> for A.

Example: The following is a Jordan canonical form.

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

There are four Jordan blocks, the first two of size 1×1, then one of size 2×2 and one of size 3×3. Concerning the eigenvalues and eigenspaces of J, we see that $\lambda_1 = 1$ has algebraic and geometric multiplicity 1, $\lambda_2 = 2$ has algebraic multiplicity 2 but geometric multiplicity 1, and $\lambda_3 = 3$ has algebraic multiplicity 4 but geometric multiplicity 2. In fact, if we solve for the eigenvectors, we find that

$$S_1 = \operatorname{span}\{e_1\}, \;\; S_2 = \operatorname{span}\{e_3\}, \;\; S_3 = \operatorname{span}\{e_2, e_5\},$$

where $e_1, \ldots, e_7$ form the standard basis of $\mathbb{R}^7$.

Example: Find a Jordan canonical form for $B = \begin{pmatrix} \lambda & 1 & 2 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$.

<u>Solution</u>: We have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and

$$S_\lambda = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}.$$

We seek M such that $M^{-1}BM = J$, that is $BM = MJ$. We see that $J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$. Let x_1, x_2, x_3 be the columns of M, and let us examine the first column of $B\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} J$: $Bx_1 = \lambda x_1$ gives here

$$x_1 = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix},$$

for some scalars c_1, c_2.

Now look at the second column: $Bx_2 = \lambda x_2 + x_1$ means $(B - \lambda\mathbb{I})x_2 = x_1$, whereby x_1 must be chosen not just in $\mathcal{N}(B - \lambda\mathbb{I}) = S_\lambda$, but also in $\mathcal{C}((B - \lambda\mathbb{I}))$. We need

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_2 = \begin{pmatrix} c_1 \\ 2c_2 \\ -c_2 \end{pmatrix},$$

which implies $c_2 = 0$ for the system to be consistent. Then we may choose $c_1 = 1$ so that $x_1 = e_1$ and $x_2 = e_2$.

Finally, $x_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ is the other eigenvector, corresponding to the size-1 Jordan block. Thus,

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix},$$

and we see that indeed $M^{-1}BM = J$.

Example: Consider now the linear, homogeneous system of ODE's with constant coefficients $\frac{dx}{dt} = Ax$, where $M^{-1}AM = J$ is a Jordan form for A. Using the change of variables $x = My$ we arrive at the system $\frac{dy}{dt} = Jy$. This system can be solved one block at a time:

$$\frac{d}{dt}\begin{pmatrix} * \\ y_{i_1} \\ \vdots \\ y_{i_k} \\ * \end{pmatrix} = \begin{pmatrix} J_i \end{pmatrix} \begin{pmatrix} * \\ y_{i_1} \\ \vdots \\ y_{i_k} \\ * \end{pmatrix},$$

where the vector $\begin{pmatrix} y_{i_1} \\ \vdots \\ y_{i_k} \end{pmatrix}$ are those components that interact with J_i (which must be a Jordan block of size k). Then,

$$\frac{d}{dt}\begin{pmatrix} y_{i_1} \\ \vdots \\ y_{i_k} \end{pmatrix} = J_i \begin{pmatrix} y_{i_1} \\ \vdots \\ y_{i_k} \end{pmatrix} = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & \dots & \lambda_i & 1 \\ 0 & 0 & 0 & \dots & \lambda_i \end{pmatrix} \begin{pmatrix} y_{i_1} \\ \vdots \\ y_{i_k} \end{pmatrix},$$

that is

$$\begin{cases} \dfrac{dy_{i_1}}{dt} = \lambda_i y_{i_1} + y_{i_2} \\ \dfrac{dy_{i_2}}{dt} = \lambda_i y_{i_2} + y_{i_3} \\ \qquad \vdots \\ \dfrac{dy_{i_k}}{dt} = \lambda_i y_{i_k}. \end{cases}$$

Starting with the last equation we obtain $y_{i_k} = c_k e^{\lambda_i t}$. Substituting back into the previous equation,

$$\frac{dy_{i_{k-1}}}{dt} = \lambda_i y_{i_{k-1}} + c_k e^{\lambda_i t},$$

which is a first order linear ODE. By elementary methods we find that

$$\begin{aligned} y_{i_{k-1}} &= e^{\lambda_i t} \left(\int c_k e^{\lambda_i t} e^{-\lambda_i t} \, dt + c_{k-1} \right) \\ &= e^{\lambda_i t} (c_k t + c_{k-1}). \end{aligned}$$

Iterating this procedure, we see that, for $1 \leq j < k$,

$$y_{i_j} = e^{\lambda_i t} \left[c_j + \sum_{l=1}^{k-j} \frac{c_{j+l}}{l} \, t^l \right].$$