MA 511, Session 34

Review

1) Let us revisit the coefficients of the characteristic polynomial. We have
$p(\lambda) = \det(A - \lambda\mathbb{I}) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0,$
and we already know that

$$a_n = (-1)^n, \quad a_{n-1} = (-1)^{n-1}\operatorname{tr} A, \quad a_0 = \det A.$$

We also know that we can factor a polynomial using its roots, the eigenvalues of A in this case. Let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues (that may be repeated and/or complex). Then, we may also write

$$(*) \qquad p(\lambda) = (-1)^n(\lambda - \lambda_1)\dots(\lambda - \lambda_n).$$

Using elementary algebra we are led to the relation $a_{n-1} = (-1)^{n-1}(\lambda_1 + \cdots + \lambda_n)$. Hence,

$$\operatorname{tr} A = \lambda_1 + \cdots + \lambda_n,$$

that is, $a_{11} + \cdots + a_{nn} = \lambda_1 + \cdots + \lambda_n$.

We can also obtain from $(*)$ a useful identity for a_0:

$$a_0 = (-1)^n(-1)^n\lambda_1\dots\lambda_n = \lambda_1\dots\lambda_n.$$

Hence,

$$\det A = \lambda_1\dots\lambda_n.$$

2) Given the symmetric matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, find an orthogonal matrix Q such that $Q^T AQ$ is diagonal.

<u>Solution</u>: The characteristic polynomial of A is

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 0 & \lambda & -\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 1-\lambda & 2-\lambda \\ 0 & \lambda & 0 \end{vmatrix} = -\lambda\,[(1-\lambda)(2-\lambda) - 2]$$

$$= -\lambda(\lambda^2 - 3\lambda) = -\lambda^2(\lambda - 3).$$

The eigenvalues of A are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 2$. Since A is Hermitian (symmetric), it is diagonalizable and necessarily $\dim S_0 = 2$. We find a pair of orthogonal eigenvectors in S_0 by solving $Ax = 0$, i.e. $x_1 + x_2 + x_3 = 0$. Similarly, we solve $(A - 3\mathbb{I})x = 0$ to find an eigenvector for $\lambda = 3$. We see that

$$S_0 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}, \quad S_3 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

We now normalize these three vectors and thus obtain the 3 columns of Q.

$$Q = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

It is now immediate to check that $Q^T AQ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

3) **Cayley-Hamilton Theorem:** Let A be a $n \times n$ matrix, and let
$p(\lambda) = \det(A - \lambda\mathbb{I}) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$,
be its characteristic polynomial. Then, $p(A) = 0$, that is, A is a *matrix* root of its characteristic polynomial. Let us see this in the 3×3 case.

To do this, let U be a unitary matrix such that $U^H AU = T$ is upper triangular with the eigenvalues of A on its main diagonal. Let us see that $p(T) = 0$. As indicated earlier, $p(\lambda) = -(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$, so that

$$p(T) = -(T - \lambda_1\mathbb{I})(T - \lambda_2\mathbb{I})(T - \lambda_3\mathbb{I}).$$

Note that

$$(T-\lambda_1\mathbb{I})(T-\lambda_2\mathbb{I})$$

$$= -\begin{pmatrix} 0 & t_{12} & t_{13} \\ 0 & \lambda_2-\lambda_1 & t_{23} \\ 0 & 0 & \lambda_3-\lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1-\lambda_2 & t_{12} & t_{13} \\ 0 & 0 & t_{23} \\ 0 & 0 & \lambda_3-\lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}$$

and thus

$$p(T) = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} (T-\lambda_3\mathbb{I})$$

$$= \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} \lambda_1-\lambda_3 & t_{12} & t_{13} \\ 0 & \lambda_2-\lambda_3 & t_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Finally, using $A^n = (UTU^H)^n = UT^nU^H$, we obtain from $p(A) = a_3A^3 + a_2A^2 + a_1A + a_0\mathbb{I}$ that $p(A) = Up(T)U^H = U0U^H = 0$, as claimed.

4) Let A be a 2×2 matrix and consider the system $\frac{du}{dt} = Au$. If A is diagonalizable, then the general solution is $u(t) = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2$, where u_1, u_2 are linearly independent eigenvectors corresponding to the (possibly equal or complex) eigenvalues λ_1, λ_2.

- The system is <u>stable</u> means that all solutions $u(t) \to 0$ as $t \to \infty$.

- The system is <u>neutrally stable</u> means that no solution is unbounded and some solutions do not go to zero as $t \to \infty$.

- The system is <u>unstable</u> means that some solutions $u(t) \to \infty$ as $t \to \infty$.

Suppose now that A is non-diagonalizable. In any case (even $n \times n$) the columns of e^{At} are linearly independent solutions of the system (e^{At} is a fundamental matrix). Now $\lambda = \lambda_1 = \lambda_2$ and let $S_\lambda = \text{span}\{u_1\}$. Then, $u(t) = e^{\lambda t} u_1$ is a non-trivial solution of the system. We need a second linearly independent one.

By Cayley-Hamilton we know $(A - \lambda \mathbb{I})^2 = 0$. Let us exploit this fact as follows: (we use the fact that

$e^{A+B} = e^A e^B$ when A and B commute)

$$\begin{aligned} e^{At} &= e^{\lambda \mathbb{I} t} e^{(A-\lambda \mathbb{I})t} \\ &= e^{\lambda t}\left[\mathbb{I} + (A-\lambda\mathbb{I})t + \frac{(A-\lambda\mathbb{I})^2 t^2}{2} + \dots\right], \end{aligned}$$

where all terms past the second are zero and, therefore,

$$e^{At} = e^{\lambda t}\left[\mathbb{I} + (A-\lambda\mathbb{I})t\right],$$

is a fundamental matrix.

Example: Solve the system $\frac{du}{dt} = \begin{pmatrix} 0 & 1 \\ -9 & 6 \end{pmatrix} u$ using matrix exponentials.

Solution: The characteristic polynomial is $p(\lambda) = -\lambda(6-\lambda) + 9 = (\lambda-3)^2$. We easily find that S_3 has dimension 1 and, therefore, A is defective. We use the formula just derived to find

$$\begin{aligned} e^{At} &= e^{3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left[\begin{pmatrix} 0 & 1 \\ -9 & 6 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] t\right] \\ &= e^{3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -3 & 1 \\ -9 & 3 \end{pmatrix} t\right] \\ &= e^{3t}\begin{pmatrix} 1-3t & t \\ -9t & 1+3t \end{pmatrix}. \end{aligned}$$

Thus, a pair of independent solutions is

$$e^{3t}\begin{pmatrix}1-3t\\-9t\end{pmatrix}, \quad e^{3t}\begin{pmatrix}t\\1+3t\end{pmatrix}.$$

A simpler basis for the solution set is

$$e^{3t}\begin{pmatrix}1\\3\end{pmatrix}, \quad e^{3t}\begin{pmatrix}t\\1+3t\end{pmatrix}.$$

5) (page 427, B3) For the matrix $B = \begin{pmatrix}0&1&2\\0&0&0\\0&0&0\end{pmatrix}$, use $Me^{Jt}M^{-1}$ to compute the matrix exponential e^{Bt}, and compare it with the power series

$$\mathbb{I} + Bt + \frac{(Bt)^2}{2} + \dots.$$

<u>Solution</u>: We have $B^2 = 0$ so that

$$\begin{aligned} e^{Bt} &= (\mathbb{I} + (A - \lambda\mathbb{I})t) \\ &= \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \begin{pmatrix}0&1&2\\0&0&0\\0&0&0\end{pmatrix} t + 0 \\ &= \begin{pmatrix}1&t&2t\\0&1&0\\0&0&1\end{pmatrix}. \end{aligned}$$

On the other hand, we know from last session that

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix},$$

gives $M^{-1}BM = J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Also, since $J^2 = 0$,

$$e^{Jt} = \mathbb{I} + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and, therefore, we find again that

$$\begin{aligned} Me^{Jt}M^{-1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & t & 2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

6) Lagrange interpolation (A. Khovanskii).

Let $P(x) = (x - \lambda_1) \cdots (x - \lambda_n)$ be a polynomial, and $Q(x)$ a function (e.g., another polynomial or $\exp(x)$).

If all λ_j are distinct, Lagrange interpolation of Q on the roots of P is a polynomial $R(x)$ of degree less than n such that $R(\lambda_j) = Q(\lambda_j)$ for all j. If Q is a polynomial, R is the remainder of division of Q by P. There is a simple formula for R:

$$R(x) = Q(\lambda_1) P_{\lambda_1}(x) + \cdots + Q(\lambda_n) P_{\lambda_n}(x),$$

$$\text{where} \quad P_\lambda(x) = \frac{P(x)}{P'(\lambda)(x - \lambda)}.$$

The polynomial $P_{\lambda_j}(x)$ equals 1 at $x = \lambda_j$ and vanishes at λ_k when $k \neq j$.

Theorem: Let $P(x)$ be the characteristic polynomial of A, or any polynomial such that $P(A) = 0$. If $Q(x)$ is a polynomial, then $R(A) = Q(A)$. If $Q(x) = \exp(xt)$ then $R(A) = \exp(At)$.

This follows from the Cayley-Hamilton Theorem.

If $P(x)$ has roots $\lambda_1, \ldots, \lambda_k$ with multiplicities $n_1, \ldots, n_k$, the theorem still holds, but Lagrange interpolation with multiplicities requires that $R(x)$ and $Q(x)$ have the same Taylor polynomial of order $n_j - 1$ at $x = \lambda_j$, for each j. This gives $n = n_1 + \cdots + n_k$ equations for the n unknown coefficients of R.

Example: Let $P(x) = (x - \lambda)^2$. Then

$$R(x) = Q(\lambda) + Q'(\lambda)(x - \lambda), \quad \text{hence}$$
$$\exp(At) = e^{\lambda t}\mathbb{I} + te^{\lambda t}(A - \lambda\mathbb{I}).$$

Example: Let $P(x) = (x-1)^2(x+1), \quad Q(x) = e^{xt}$. Find $R(x) = ax^2 + bx + c$.

$$R(1) = a + b + c = Q(1) = e^t$$
$$R'(1) = 2a + b = Q'(1) = te^t$$
$$R(-1) = a - b + c = Q(-1) = e^{-t}$$

Hence $b = (e^t - e^{-t})/2$, $a = te^t/2 - (e^t - e^{-t})/4$, $c = 3e^t/4 + e^{-t}/4 - te^t/2$. If P is a characteristic polynomial of A, then $\exp(At) =$

$$\frac{2te^t - e^t + e^{-t}}{4}A^2 + \frac{e^t - e^{-t}}{2}A + \frac{3e^t + e^{-t} - 2te^t}{4}\mathbb{I}.$$