MA 511, Session 36

Quadratic Forms

In multivariate calculus, an important problem is to find maxima and minima of a function of several variables. Let us consider the case where the function F depends on 2 variables:

$$z = F(x, y).$$

We want to understand the condition(s) under which F has a local extreme value at (x_0, y_0). By considering $F(x + x_0, y + y_0)$ we may take this point to be $(0, 0)$. Then, by replacing F by $F - F(0, 0)$ we may assume $F(0, 0) = 0$.

In multivariate calculus we also study Taylor series in several variables.

Suppose $F(x, y)$ has a Taylor series at $(0, 0)$:

$$F(x, y) = a_{10}\, x + a_{01}\, y + a_{20}\, x^2 + a_{11}\, xy + a_{02}\, y^2 \\ + \text{ higher order tems}$$

Then, $a_{10} = F_x(0, 0), \quad a_{01} = F_y(0, 0),$

$$a_{20} = \frac{F_{xx}(0, 0)}{2}, \quad a_{11} = F_{xy}(0, 0), \quad a_{02} = \frac{F_{yy}(0, 0)}{2}.$$

The higher order terms involve products of powers of x and y so they go to 0 faster near $(0,0)$ and usually do not influence the local behavior of F (we'll see later when the higher order terms cannot be neglected).

Now, if $(0,0)$ is a candidate for a local extreme value (i.e. a *critical point*), then F_x and F_y are 0 at the point, i.e. $a_{10} = a_{01} = 0$. Thus, the problem of understanding if F has a local maximum or minimum (or neither) comes down to understanding the quadratic terms alone.

For this reason we shall focus on <u>quadratic forms</u>

$$f(x,y) = ax^2 + 2bxy + cy^2.$$

Some possibilities are

- A local minimum; in this case $f(x,y) > 0$ for all $(x,y) \neq (0,0)$ and the quadratic form is <u>positive definite</u>.

- A local maximum; in this case $f(x,y) < 0$ for all $(x,y) \neq (0,0)$ and the quadratic form is <u>negative definite</u>.

- A saddle point; in this case $f(x,y) < 0$ for some (x,y) and $f(x,y)>0$ for others; the quadratic form is <u>indefinite</u>.

We can write the quadratic form using matrices and vectors:

$$f(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This is a <u>2-dimensional quadratic form</u>.

Conversely, given a $n \times n$ matrix A, we can define a <u>n-dimensional quadratic form</u> by

$$f(x) = x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j, \quad x \in \mathbb{R}^n.$$

The central problem is now to determine conditions on A which tell us that the form is positive definite, i.e.

$$x^T A x > 0 \ \text{ for any } \ x \neq 0.$$

Let us examine first the case $n=2$. First note that $a=0$ gives $f(1,0)=0$ so that f cannot be positive definite if $a=0$. Completing squares ($a \neq 0$),

$$f(x,y) = a\left(x + \tfrac{b}{a}\, y\right)^2 + \left(c - \tfrac{b^2}{a}\right) y^2.$$

From here we can read necessary and sufficient conditions:

$$f \text{ is positive definite } \Leftrightarrow a > 0 \text{ and } ac - b^2 > 0.$$

This gives the familiar *second derivative test* for $F(x, y)$ in multivariate calculus when $F_x = 0$ and $F_y = 0$ at the point. If

$$F_{xx} > 0 \text{ and } F_{xx}F_{yy} - F_{xy}^2 > 0,$$

then F has a local minimum at the point. Note also that, if $F_{xx} < 0$ and $F_{xx}F_{yy} - F_{xy}^2 > 0$, then we have a local maximum and, if $F_{xx}F_{yy} - F_{xy}^2 < 0$, then it is a saddle point. In case $F_{xx}F_{yy} - F_{xy}^2 = 0$ we cannot decide what kind of a point (x_0, y_0) is (it depends on the higher order terms in the Taylor series of F).

Returning now to the $n \times n$ case, suppose that

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = x^T A x,$$

where $A = (a_{ij})$. Since symmetric matrices have many useful properties, we would like to make A symmetric. This is accomplished by replacing a_{ij} and a_{ji} by their average $\frac{a_{ij}+a_{ji}}{2}$. This change does not alter the form f at all since

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i=1}^{n} \sum_{j<i} (a_{ij} + a_{ji}) x_i x_j.$$

Thus, we may represent <u>any</u> quadratic real form f by $x^T Ax$, where A is a symmetric matrix.

Example: Decide what kind of critical point $(2,0)$ is for $F(x,y) = (x^2 - 4x)(e^y - y)$.

<u>Solution</u>: First we check that $(2,0)$ is indeed a critical point:

$$F_x(2,0) = (2x-4)(e^y - y)|_{(2,0)} = 0,$$
$$F_y(2,0) = (x^2 - 4x)(e^y - 1)|_{(2,0)} = 0.$$

Now, we also see that

$$F_{xx}(2,0) = 2(e^y - y)|_{(2,0)} = 2,$$

$$F_{yy}(2,0) = (x^2 - 4x)e^y|_{(2,0)} = -4$$

$$F_{xy}(2,0) = (2x-4)(e^y - 1)|_{(2,0)} = 0,$$

so that $F_{xx}F_{yy} - F_{xy}^2 < 0$ at $(2,0)$ and thus it is a saddle point.