MA 511, Session 37

Quadratic Forms and Quadrics

Consider the general quadratic form in n variables

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = x^T A x,$$

where $A = (a_{ij})$.

Example: Let

$$f(x_1, x_2, x_3) = x^T \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} x$$
$$= x_1^2 + 2x_1x_2 + 3x_1x_3 + 4x_2x_1 + 5x_2^2$$
$$+ 6x_2x_3 + 7x_3x_1 + 8x_3x_2 + 9x_3^2.$$

The matrix A that represents the quadratic form $f(x) = x^T A x$ is not unique. We <u>always</u> take the (unique) symmetric one obtained by replacing a_{ij} and a_{ji} by their average. In the example,

$$f(x) = x^T \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix} x$$

as well.

Definition: The quadratic form is

(i) positive definite if $f(x) > 0$ for $x \neq 0$;

(ii) positive semi-definite if $f(x) \geq 0$ for $x \neq 0$;

(iii) negative definite if $f(x) < 0$ for $x \neq 0$;

(iv) negative semi-definite if $f(x) \leq 0$ for $x \neq 0$;

(v) indefinite if $f(x) > 0$ for some x and $f(x) < 0$ for some x.

Theorem: The following statements are equivalent for a real symmetric matrix.

(i) A is positive definite;

(ii) All eigenvalues λ_k of A are positive.

(iii) All upper left submatrices A_k of A have positive determinants.

(iv) All pivots d_k in the Gaussian elimination (without row exchanges) are positive.

(v) There is a matrix R with linearly independent columns such that $A = R^T R$.

Proof: Let us show that (i) $\Leftrightarrow$ (ii):

We assume first that $x^T A x > 0$ for all $x \neq 0$. Let v_j be an eigenvector of A with the eigenvalue λ_j. Then $v_j^T A v_j = \lambda_j ||v_j||^2 > 0$, hence $\lambda_j > 0$.

Conversely, assume that all λ_k are positive. Let $Q^T A Q = \Lambda$ be diagonal, Q orthogonal, and let $y = Q^T x$. Then $x \neq 0$ implies $y \neq 0$. Hence

$$x^T A x = y^T \Lambda y = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 > 0.$$

Now we show that (ii) $\Rightarrow$ (v):

Assume $Q^T A Q = \Lambda$ is diagonal and let

$$\sqrt{\Lambda} = \begin{pmatrix} \sqrt{\lambda_1} & 0 \ldots 0 & & \\ 0 & \sqrt{\lambda_2} & \ldots & 0 \\ & & \vdots & \\ 0 & 0 & \ldots & \sqrt{\lambda_n} \end{pmatrix}, \quad R = \sqrt{\Lambda}\, Q^T.$$

Then, we can readily see that $(\sqrt{\Lambda})^2 = \Lambda$, and thus $A = Q \Lambda Q^T = (Q\sqrt{\Lambda})(\sqrt{\Lambda} Q^T) = R^T R$, as needed.

Finally we show that (v) $\Rightarrow$ (i):

$A = R^T R$ is symmetric, since $A^T = (R^T R)^T = R^T R = A$. Since R is invertible (full rank), we know that given $x \in \mathbb{R}^n$, $x = 0 \Leftrightarrow Rx = 0$. Thus, $x^T A x = x^T R^T R x = (Rx)^T (Rx) = \|Rx\|^2 > 0$ unless $Rx = 0$, that is unless $x = 0$.

Remark: It follows from (ii) that positive definite matrices have positive determinants.

(iii) and (iv) are equivalent since $\det A_k = d_1 \cdots d_k$. (iii) follows from (ii) applied to A_k. Note that A_k is positive definite when A is positive definite. Conversely, if (iv) holds, then $A = LDL^T$ (see p.51) where L is lower diagonal with 1's on the diagonal, and D is the pivot matrix. Let $L_t = tL + (1-t)\mathbb{I}$ be a line segment in the space of lower diagonal matrices with 1's on the diagonal connecting $\mathbb{I}$ and L. Then $A_t = L_t D L_t^T$ is symmetric and $\det A_t = \det D > 0$. All eigenvalues of $A_0 = D$ are positive, and A_t cannot have a zero eigenvalue. Hence eigenvalues cannot change sign and should remain positive for any A_t, including $A_1 = A$.

We can simplify the quadratic form very noticeably by introducing an appropriate change of variables. Since A is symmetric there is an orthogonal matrix Q such that $Q^TAQ = \Lambda$ is diagonal. Let $y = Q^Tx$. Then, since $A = Q\Lambda Q^T$, we have

$$x^TAx = x^TQ\Lambda Q^Tx = y^T\Lambda y = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2,$$

This is a pure quadratic form, i.e. one without any cross product terms.

Consider now the locus of points in $\mathbb{R}^n$ such that $f(x) = x^TAx = a^2$. Recall that orthogonal matrices geometrically preserve angles and distances from the origin. Since the locus of $\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 = a^2$ $(a \neq 0)$ is an ellipsoid (an ellipse if $n = 2$), then so is the original locus $f(x) = x^TAx = a^2$.

Example: Give a geometric description of the ellipse

$$f(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1.$$

Solution: $p(\lambda) = (5-\lambda)^2 - 9 = \lambda^2 - 10\lambda + 16$ $= (\lambda - 2)(\lambda - 8)$. So, the eigenvalues of A are $\lambda_1 = 2$

and $\lambda_2 = 8$. We find unit eigenvectors v_1 and v_2 as solutions of, respectively,

$$\begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, $\quad v_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, for example.

Define

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \ \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}, \ A = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}.$$

It then follows that $Q^T A Q = \Lambda$. With the change of variables $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = Q^T x$, we see that the given ellipse, the locus of points $y \in \mathbb{R}^2$ such that $y^T \Lambda y = 2y_1^2 + 8y_2^2 = 1 = x^T A x$, is the ellipse with major semiaxis $a = \frac{\sqrt{2}}{2}$ in the direction of v_1 and minor semiaxis $b = \frac{\sqrt{2}}{4}$ in the (orthogonal) direction of v_2. Q represents a counterclockwise rotation by $\frac{\pi}{4}$ from the canonical position, that is $Q = R_{\frac{\pi}{4}}$.