# MA 511, Session 38

## The Finite Element Method

We have been studying positive definite forms
$F(x) = x^T Ax = (x, Ax), \quad x \in \mathbb{R}^n, \quad A$ symmetric.
We have seen that in abstract vector spaces we have linear operators and scalar products. Thus, if $V$ is an abstract vector space and $L : V \longrightarrow V$ a linear operator, we say a quadratic form on $V$ is

$$F(v) = \big(v, L(v)\big).$$

We say the form is positive if

$$\big(v, L(v)\big) > 0 \text{ for all } v \in V,\ v \neq 0.$$

$L$ is symmetric (or self-adjoint) if

$$(u, L(v)) = (L(u), v) \text{ for all } u, v \in V.$$

**Example:** An important example of this in physics is the following. Let $\mathcal{C}_0^2(D)$ be the vector space of functions having two continuous derivatives on a bounded domain $D$ and taking on the value zero on the boundary of the domain, $\partial D$. Then, the operator

$$-\Delta v = -\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

(called minus the Laplacian) is a positive self-adjoint operator (in analogy with a positive definite symmetric matrix). Here the inner product is

$$(u,v) = \iint_D u(x,y)\, v(x,y)\, dx\, dy.$$

By Green's theorem in multivariate calculus,

$$\iint_D (P_x + Q_y)\, dx\, dy = \int_{\partial D} (P\, dy - Q\, dx),$$

applied to $P = u\, v_x$ and $Q = u\, v_y$,

$$(u, -\Delta v) = -\iint_D u\, \Delta v\, dx\, dy = \iint_D (u_x v_x + u_y v_y)\, dx\, dy.$$

Hence $-\Delta$ is self-adjoint, $(u, -\Delta v) = (-\Delta u, v)$, and

$$F(v) = (v, -\Delta v) = \iint_D |\nabla v|^2\, dx\, dy,$$

is positive unless $v$ is the zero function.

To keep things simple, we shall consider the one dimensional version only. Let $D = [0,1]$ and $L(v) = -\frac{d^2 v}{dx^2}$ be defined on
$V = \{v : D \longrightarrow D, v \in \mathcal{C}^2(D) \text{ and } v(0) = v(1) = 0\}$,
so that

$$F(v) = -\int_0^1 v(x)\, v''(x)\, dx = \int_0^1 v'(x)^2\, dx > 0$$

unless $v \equiv 0$.

We want to show now that the problem of solving

$$(*) \qquad Ax = b, \quad A \text{ symmetric},$$

(or, in an abstract setting, $-\Delta u = f(x,y)$ or $-\frac{d^2u}{dx^2} = f(x)$) is related to the seemingly unrelated problem of finding the vector $x$ for which the quadratic form

$$P(x) = \tfrac{1}{2}x^TAx - x^Tb \text{ is minimized}$$

(or, in an abstract setting,

$$\iint\limits_D \Big(-\frac{1}{2}u(x,y)\Delta u(x,y) - u(x,y)f(x,y)\Big)\,dx\,dy$$

or

$$(**) \qquad \int_0^1 \big(-\tfrac{1}{2}u(x)\,u''(x) - u(x)\,f(x)\big)\,dx$$

are minimized). The connection between the two problems is given by the following.

**Theorem:** Let $L$ be a self-adjoint linear transformation of $V$, and $\big(x, L(x)\big)$ a positive form. Then, for $f \in V$,

$$P(x) = \tfrac{1}{2}\big(x, L(x)\big) - (x, f)$$

has its minimum when

$$L(x) = f.$$

Proof: Suppose $L(x) = f$. Then, for any $y \in V$,

$$P(y) - P(x) = \tfrac{1}{2}\big(y, L(y)\big) - (y, f) - \tfrac{1}{2}\big(x, L(x)\big) + (x, f)$$
$$= \tfrac{1}{2}\big(y, L(y)\big) - \big(y, L(x)\big) + \tfrac{1}{2}\big(x, L(x)\big)$$
$$= \tfrac{1}{2}\big((y - x), L(y - x)\big) > 0$$

unless $y = x$. Here we used $(x, L(y)) = (L(x), y)$.

Simple Approximation Problem:

Consider the 2-point boundary value problem $-u'' = f$ in the interval $[0, 1]$, with $u(0) = u(1) = 0$. Instead of finding the exact solution in $V = \mathcal{C}_0^2[0, 1]$, we seek an approximate solution in a finite dimensional subspace $W$. Let $\{v_1, \ldots, v_n\}$ be a basis of $W$. Let $y_1, \ldots, y_n$ be unknown scalars to minimize $(**)$ for $u = y_1 v_1 + \cdots + y_n v_n \in W$ rather than $u \in V$. Note that

$$P(u) = \tfrac{1}{2} \int_0^1 (y_1 v_1' + \cdots + y_n v_n')^2 \, dx$$
$$- \int_0^1 (y_1 v_1 + \cdots + y_n v_n) \, f(x) \, dx.$$

Let

$$a_{ij} = \int_0^1 v_i'(x)\, v_j'(x)\, dx, \qquad b_j = \int_0^1 v_j(x)\, f(x)\, dx,$$

and define the $n \times n$ matrix $A = (a_{ij})$ and the vectors $b, y \in \mathbb{R}^n$ with components $b_1, \ldots, b_n$ and $y_1, \ldots, y_n$, respectively. Then, the minimum of $P$ for $u \in W$ corresponds to the minimum of the quadratic form $\frac{1}{2} y^T A y - y^T b$ for $y \in \mathbb{R}^n$ which, by our theorem is minimized by the solution of the linear system $Ay = b$.

**Example:** (Finite Element Method) Let $n$ be a positive integer and consider the partition of $[0, 1]$ into $n$ uniform subintervals $I_j = [x_{j-1}, x_j]$ $(1 \le j \le n)$ of length $h = \frac{1}{n}$ given by the nodes $x_k = kh$ for $0 \le k \le n$. For $1 \le j \le n-1$ define now the "roof top" functions

$$v_j(x) = \begin{cases} \frac{1}{h}(x - x_{j-1}), & x_{j-1} \le x \le x_j, \\ \frac{1}{h}(x_{j+1} - x), & x_j \le x \le x_{j+1}, \\ 0 & \text{elsewhere.} \end{cases}$$

These functions are linear on each subinterval, continuous over the whole interval $[0,1]$, and they take the value 1 at the node with the same index as the function and the value 0 at all other nodes, that is $v_j(x_k) = \delta_{jk}$. Note that the slopes of the line segments in their graphs are equal to $\pm n$. Then, integrating the products that define the coefficients $a_{ij}$, we find that $A$ is tridiagonal,

$$a_{ij} = \begin{cases} \displaystyle\int_{x_{i-1}}^{x_i} n^2\,dx + \int_{x_i}^{x_{i+1}} (-n)^2\,dx = 2n, & i = j, \\ \displaystyle\int_{\min\{x_i,x_j\}}^{\max\{x_i,x_j\}} (n)(-n)\,dx = -n, & |i-j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let us choose now $f(x) = -x^2$ and $n = 4$. Then,

$$b = \begin{pmatrix} 4\displaystyle\int_0^{\frac{1}{4}} x^3\,dx - 4\int_{\frac{1}{4}}^{\frac{1}{2}} (x-\tfrac{1}{2})x^2\,dx \\ 4\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} (x-\tfrac{1}{4})x^2\,dx - 4\int_{\frac{1}{2}}^{\frac{3}{4}} (x-\tfrac{3}{4})x^2\,dx \\ 4\displaystyle\int_{\frac{1}{2}}^{\frac{3}{4}} (x-\tfrac{1}{2})x^2\,dx - 4\int_{\frac{3}{4}}^{1} (x-1)x^2\,dx \end{pmatrix} = \begin{pmatrix} -\frac{1}{96} \\ -\frac{1}{48} \\ -\frac{1}{32} \end{pmatrix},$$

and the approximation is $y_1 v_1(x) + y_2 v_2(x) + y_3 v_3(x)$, where $y \in \mathbb{R}^3$ is the solution of

$$\begin{pmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{96} \\ -\frac{1}{48} \\ -\frac{1}{32} \end{pmatrix} = -\frac{1}{96} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

The exact solution is $u(x) = \frac{1}{12}\, x^4 - \frac{1}{12}\, x$, and the approximation is its (orthogonal) projection into the 3-dimensional subspace of $V$ spanned by $v_1, v_2, v_3$. The solution of this linear system is

$$y = -\frac{1}{768} \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}.$$